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On the structure and properties of lower bounded analytic frames

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Abstract

Algebraic frames are generalizations of Fourier transforms on locally compact abelian groups in the sense that the family of vectors forming the frame are replaced by a family of unbounded linear functionals. The paper studies the indexing measure space of the algebraic frames; as the investigation narrows down to the class of lower semi-frames, more relations are revealed between the discreteness and the redundancy of the frame family.

Keywords: Analysis operator; generalized frame; lower semi-frame; redundancy; discreteness; algebraic frames

1. Introduction

The notion of an algebraic frame in a separable Hilbert space was defined in [1] as an extension of the various notions of frame, generalized frame and upper/lower semi-frame ([1-15]) in which a family of vectors in H, viewed as bounded linear functionals on H, are replaced by a family of unbounded linear functionals. By a *pseudo-frame* on a separable Hilbert space H, we mean a family $\Theta = (\theta_z)_{z\in Z} \subset X'$ indexed by a (positive) measure space $(Z, \mathcal{M}, \mu)$ in which X' is the algebraic dual of a dense linear subspace X of H and, for all $h \in X$, the function $\tilde{h}$ defined by $\tilde{h}(z) := \langle h, \theta_z \rangle$ is μ-square summable. (The function $\tilde{h}$ may be viewed as an element of $\mathbb{C}^Z$ and/or $L^2(\mu)$ as understood from the context.) The space X, denoted by $\mathcal{D}(\Theta)$, may be called the domain of Θ. Let $t_\Theta: \mathcal{D}(\Theta) \to L^2(\mu)$ be the linear operator defined by $t_\Theta h = \tilde{h}$. If t_Θ is closable, its closure, denoted by T_Θ, is called the *analysis operator* of Θ. Note that the domain $\mathcal{D}(T_\Theta)$ of T_Θ contains $\mathcal{D}(\Theta)$ as a dense subspace. However, one can use a Hausdorff maximality argument to assume without loss of generality that $\mathcal{D}(\Theta) = \mathcal{D}(T_\Theta)$. To see this, let $\mathcal{S}$ denote the class of all linear subspaces $\mathcal{R}$ of the linear space $\mathbb{C}^Z$ with the following properties:

(i) $\tilde{h} \in \mathcal{R}$ for all $h \in \mathcal{D}(\Theta)$; (ii) each $f \in \mathcal{R}$ represents an element of $\mathcal{R}(T_\Theta)$; and (iii) no distinct pair of elements of $\mathcal{R}$ are equal a.e. $[\mu]$. Now, the union of any chain in $\mathcal{S}$ falls in $\mathcal{S}$ and, hence, $\mathcal{S}$ contains a maximal element $\mathcal{R}_0$. Now, it follows from the maximality of $\mathcal{R}_0$ that, for all $h \in \mathcal{D}(T_\Theta)$, the $L^2(\mu)$-function $T_\Theta h$ is uniquely represented by an element $f_h \in \mathcal{R}_0$; hence, one can assume $\mathcal{D}(\Theta) = \mathcal{D}(T_\Theta)$ by defining $\langle h, \theta_z \rangle = f_h(z)$ for all $z \in Z$.

A pseudo-frame with an injective analysis operator is called an *algebraic frame*. An algebraic frame is called *lower* (resp. *upper*) *bounded* if there exists a positive constant c such that $||T_\Theta h|| \geq c||h||$ (resp. $||T_\Theta h|| \leq c||h||$) for all $h \in \mathcal{D}(t_\Theta)$. An *analytic frame* is an algebraic frame in which all linear functionals θ_z are bounded. In light of the Riesz representation, the analytic frame Θ can be identified with a subset of the Hilbert space H, which is how the frames, generalized frames, etc. are defined. Note that, if X is the collection of all simple functions in $L^2(\mathbb{R}^n)$, then the linear functionals $\theta_z \in X'$ defined by

$$\langle h, \theta_z \rangle = \int_{\mathbb{R}^n} h(t) e^{-it.z} \ (\forall z \in \mathbb{R}^n)$$

yield a bilaterally bounded algebraic frame whose analysis operator is the Fourier transform from $L^2(\mathbb{R}^n)$ onto $L^2(\mathbb{R}^n)$; although its analysis operator is unitary, it is not analytic, anyway. In general, if $\mathcal{R}(T_\Theta) \neq L^2(\mu)$, then Θ is called redundant.

If Θ is a pseudo-frame consisting of bounded linear functionals, then it follows from the proof of Lemma 2.1 of [2] that the corresponding t_Θ is closable; in fact, if $(h_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\Theta)$ converges to 0 and the sequence $\widetilde{h_n}$ converges to f in $L^2(\mu)$-norm, then a subsequence of $\widetilde{h_n}$ converges pointwise to f a.e. $[\mu]$; hence, $f = \tilde{0} = 0$ a.e. $[\mu]$. Therefore, the desired closure T_Θ exists and it is easy to see that

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$$\mathcal{D}(T_\Theta) \subset \{h \in H: \int_Z |\langle h, \theta_z\rangle|^2 d\mu(z) < \infty\}. \quad (1)$$

Upper bounded analytic frames Θ are the same as upper semi-frames defined in [2]; in this case the inclusion in (1) will become an equality and, more precisely, $\mathcal{D}(T_\Theta) = H$. Our class of lower bounded analytic frames maybe different from that of lower semi-frames [2] in various aspects. The first difference is the density of $\mathcal{D}(T_\Theta)$; for algebraic frames we assume the initial domain $\mathcal{D}(\Theta)$ and, hence, the resulting domain $\mathcal{D}(T_\Theta)$ is dense in the underlying Hilbert space H, while for lower semi-frames, the domain is the set on the right-hand side of the inclusion (1). Then, to avoid pathological problems, conditions are imposed to guarantee the density of $\mathcal{D}(T_\Theta)$. (For example, see Lemma 2.3 of [2].) The second difference is in the analysis operator which is automatically closed for analytic frames but has to be assumed so for algebraic frames. Finally, for a given pair of lower bounded analytic frames indexed by the same measure space, we do not know of the equality of their analysis operators when the domain of one is included in the domain of the other. For this reason, we regard the class of densely defined lower semi-frames as a proper subset of the class of lower bounded analytic frames.

In Section 2, we will study the structure of the measure space indexing an algebraic frame and, in case we have a lower bounded analytic frame, we prove a strong relation between the frame, its discreteness and its redundancy. The paper is concluded by examples of lower bounded pseudo-frames coming short of closed analysis operators.

2. The structure of algebraic or analytic frames

Throughout the remainder of the paper, $\Theta = (\theta_z)_{z\in Z}$ denotes an algebraic frame in an infinite-dimensional separable Hilbert space H indexed by a measure space $(Z, \mathcal{M}, \mu)$.

In what follows we will need the notion of isometric equivalence of two algebraic frames characterized in the following definition and lemma.

Definition 1. The pseudo-frames $\Theta = (\theta_z)_{z\in Z}$ and $\Xi = (\xi_y)_{y\in Y}$ on a common signal space $\mathcal{D}(\Theta) = \mathcal{D}(\Xi)$ are said to be isometric if

$$\int_Z |\langle \theta_z, h\rangle|^2 d\mu(z) = \int_Y |\langle \xi_y, h\rangle|^2 d\nu(y)$$

for all $h \in X$, where μ and ν are the corresponding measures.

Lemma 1. Assume $\Theta = (\theta_z)_{z\in Z}$ is an algebraic frame isometric to a pseudo-frame $\Xi = (\xi_y)_{y\in Y}$ on a common signal space $\mathcal{D}(\Theta) = \mathcal{D}(\Xi)$. Then Ξ is an algebraic frame and $\mathcal{D}(T_\Theta) = \mathcal{D}(T_\Xi)$. Moreover, $\mathcal{R}(T_\Theta)$ is closed in $L^2(\mu)$ if and only if $\mathcal{R}(T_\Xi)$ is closed in $L^2(\nu)$.

Proof: Define $U: T_\Theta X \subset L^2(\mu) \to L^2(\nu)$ by $(UT_\Theta h)(y) = \langle h, \xi_y\rangle$ for all $h \in \mathcal{D}(\Theta)$ and all $y \in Y$, where μ and ν are the corresponding measures on Z and Y, respectively. Since U is an isometry, it has an isometric extension to the closure of $T_\Theta\, \mathcal{D}(\Theta)$ which is denoted by the same letter U. It is easy to see that the operator $UT_\Theta: \mathcal{D}(T_\Theta) \to L^2(\nu)$ is injective and is the closure of the operator sending $h \in \mathcal{D}(\Theta)$ to the function $y \mapsto \langle h, \xi_y\rangle$ in $L^2(\nu)$. The last part of the lemma follows from the fact that an isometry maps complete subspaces onto complete ones.

The following theorem simplifies the measure theoretic structure of an algebraic frame under isometric modifications.

Theorem 1. Let $\Theta = (\theta_z)_{z\in Z}$ be an algebraic frame indexed by the measure space $(Z, \mathcal{M}, \mu)$ with completion $(Z, \overline{\mathcal{M}}, \bar{\mu})$. Then the following assertions are true.
(1). There exists a σ-finite μ -measurable set $Y \subset \{z \in Z: \theta_z \neq 0\}$ such that the family $\Xi = (\theta_z)_{z\in Y}$ indexed by the measure space $(Y, \mathcal{N}, \nu)$ is an algebraic frame isometric to Θ, where $\mathcal{N} = Y \cap \overline{\mathcal{M}}$ and $\nu = \bar{\mu}|_{\mathcal{N}}$.
(2). Let $\Xi, Y, \mathcal{N}, \nu$ be as in the previous part. Then there exists an equivalence relation on Y with the quotient map φ such that the algebraic frame Ξ is isometric to an algebraic frame $\hat{\Theta}$ indexed by the measure space $(\varphi(Y), \varphi(\mathcal{N}), \nu\, o\varphi^{-1})$ whose atoms are singleton and whose singleton sets are measurable.

Proof: Since T_Θ is closed and densely defined, the polar decomposition $T_\Theta = V|T_\Theta|$ exists and, hence, $\mathcal{R}(T_\Theta) \subset VH$ is separable. Let $(f_n)_{n\in\mathbb{N}}$ be a countable dense subset of $\mathcal{R}(T_\Theta)$ and let $\Gamma_n = \{z \in Z: f_n(z) \neq 0\}$. Since each f_n is square integrable, it follows that the corresponding Γ_n and, hence, the sets $Y = \cup_{n=1}^{\infty} \Gamma_n$ are σ-finite μ-measurable subsets of Z. Thus, the sequence $(f_n)_{n\in\mathbb{N}}$ consists of functions which are zero almost everywhere outside Y. Since every function in $\overline{R(T_\Theta)}$ is the almost everywhere limit of a subsequence of $(f_n)_{n\in\mathbb{N}}$, it follows that $\langle h, \theta_z\rangle$=0 for almost all $z \in Z \setminus\circledcirc Y$. Now, let $(Y, \mathcal{N}, \nu)$ be as in part (1) and observe that, for all $h \in \mathcal{D}(t_\Theta)$,

$$\int_Z |\langle h,\theta_z\rangle|^2 d\mu(z) = \int_Y |\langle h,\theta_y\rangle|^2 d\nu(y).$$

This proves part (1).

For part (2), let $Y = \cup_{k\in\mathbb{N}} Y_k$ be a partition of Y into sets of finite measures and let $\{\Delta_i: i \in I\}$ be the family of all atoms of ν. Let I_k be the set of those $i \in I$ such that $\nu(Y_k \cap \Delta_i) \neq 0$. By definition, $I_k \cap I_j = \emptyset$ for all $k \neq j$. Since $\sum_{i\in I_k} \nu(\Delta_i) \leq \nu(Y_k) < \infty$, it follows that I_k is countable and, hence, I can be identified as $\mathbb{N}$ or as a segment $\{1,2,\ldots,n\}$ of $\mathbb{N}$. Now, by ignoring a set of measure zero, one can assume without loss of generality that $\Delta_i \cap \Delta_j = \emptyset$ for all distinct $i,j \in I$; define $Y_0 = Y \circledcirc \cup_{i\in I} \Delta_i$. For $y,z \in Y$, define $y \sim z$ if $y = z \in Y_0$ or $y,z \in \Delta_i$ for some $i \in I$. We thus obtain an equivalence relation whose quotient map will be denoted by φ. Identifying $\varphi(Y_0)$ by Y_0, the quotient space $\varphi(Y)$ can be partitioned as $\varphi(Y) = Y_0 \dot\cup \{\varphi(\Delta_i)\colon i \in I\}$. Now, define $\varphi(\mathcal{M}) := \{\varphi(M): M \in \mathcal{M}\}$ and $\omega(M) := (\mu\, o\varphi^{-1})(\varphi(N)) = \mu(\varphi^{-1}(\varphi(N)))$ for all $N \in \mathcal{N}$. Clearly, Ω is a σ-finite measure on (the σ-algebra) $\varphi(\mathcal{N})$ having the singletons $\{\{\varphi(\Delta_i)\}: i \in I\}$ as its collection of all atoms. Moreover, $\omega(\{\varphi(\Delta_i)\}) = \mu(\Delta_i)$ for all $i \in K$ and ω and ν coincide on Y_0.

Now, for $h \in \mathcal{D}(\Theta)$, observe that the complex-valued function $\langle h,\theta_y\rangle$ defined on Y is essentially constant on each Δ_i. Assume without loss of generality that, for each $h \in \mathcal{D}(\Theta)$, the function $\langle h,\theta_y\rangle$ is constant on each Δ_i. Define $\widetilde{\Theta} := (\widetilde{\theta_\zeta})_{\zeta\in\varphi(Y)}$ by $\langle h,\widetilde{\theta_\zeta}\rangle = \langle h,\theta_z\rangle$ for any $z \in \varphi^{-1}(\zeta)$. It now follows that, for every $h \in \mathcal{D}(\Theta)$,

$$||T_{\widetilde{\Theta}} h||^2 = \int_{\varphi(Y)} |\langle h,\widetilde{\theta_\zeta}\rangle|^2\, d\omega(\zeta)$$
$$= \int_{Y_0} |\langle\theta_z,h\rangle|^2\, d\nu(z) + \sum_{i\in I} |\langle h,\theta_{z_i}\rangle|^2 \nu(\Delta_i) = ||T_\Xi\, h||^2,$$

where $z_i \in \Delta_i$ is any representative of the equivalence class Δ_i $(i \in I)$.

The following theorem studies the structure of analytic frames. A special case of Part (2) of the theorem was proved in [5] for generalized frames with surjective analysis operators. In the following, by a *normalized* analytic frame, $\Theta = (\theta_z)_{z\in Z}$, we mean one for which $||\theta_z||=1$ for almost all z.

Theorem 2. Let $\Theta = (\theta_z)_{z\in Z}$ be an analytic frame indexed by $(Z,\mathcal{M},\mu)$. The following assertions are true.

(1). Θ is isometric to a normalized analytic frame such that the corresponding measures have the same collection of atoms. Moreover, if Θ is normalized, then the operator $T_\Theta: \mathcal{D}(T_\Theta) \to L^\infty(\mu)$ has a (continuous) extension to H sending the unit ball of H to the unit ball of $L^\infty(\mu)$.

(2). If $\Theta = (\theta_z)_{z\in Z}$ is a lower bounded analytic frame, if $\theta_z \neq 0$ for almost all z and if $\mathcal{R}(T_\Theta)$ is a closed subspace with a finite co-dimension in $L^2(\mu)$, then the measure μ is discrete in the sense that Z is the union of (countably many) atoms.

Proof: In the definition of Y in Theorem 1(1), choose $f_n = T_\Theta\, e_n$ for some orthonormal basis $\{e_1,e_2,\ldots\}$ of $\mathcal{D}(t_\Theta)$. Observe that $Y = \{z \in Z: \theta_z \neq 0\}$ and that $||\theta_z||^2 = \sum_n |\langle e_n,\theta_z\rangle|^2$, where the terms of the series on the right-hand side are measurable functions. Thus, the function $||\theta_z||$ is measurable. In view of Theorem 1, assume without loss of generality that $Y = Z$ and μ is complete.
Define the normalized frame $\Omega = (\omega_z)_{z\in Z}$ by $\omega_z = \theta_z\,/\,||\theta_z||$ indexed by the measure space $(Z,\mathcal{M},\eta)$, where $d\eta(z) = ||\theta_z||^2 d\mu(z)$ for $z \in Z$. It follows that, for all $h \in \mathcal{D}(\Theta)$,

$$||T_\Omega\, h||^2 = \int_Z |\langle h,\omega_z\rangle|^2 d\eta(z) \int_Z |\langle h,\theta_z\rangle|^2 d\mu(z)$$
$$= ||T_\Theta h||^2.$$

This proves that Ω is an analytic frame isometric to Θ. Since μ and η are absolutely continuous with respect to each other, they have the same collections of atoms. Hence, one is discrete if and only if the other is so.

To complete the proof of (1), assume $||\theta_z|| = 1$ a.e., and observe that, for all $h \in \mathcal{D}(\Theta)$, $|(T_\Theta h)(z)| = |\langle h,\theta_z\rangle| \leq ||h||$ for almost all $z \in Z$. Therefore, T_Θ maps the unit ball of H into the unit ball of $L^\infty(\mu)$, and hence, the linear operator $T_\Theta: H \to L^\infty(\mu)$ has a bounded extension of norm at most 1 to all of H.

For (2), observe that Θ is isometric with a normalized lower bounded analytic frame $\Omega = (\theta_y/||\theta_y||)_{y\in Y}$ indexed by the complete measure space $(Y,\mathcal{N},||\theta_z||^2 d\mu)$. By Lemma 1, the range of the analysis operator of the latter family remains closed; we show that its co-dimension also remains finite. Let $\{g_i: i \in I\}$ be an orthonormal set in $\mathcal{R}(T_\Theta)^\perp$. Then, for $i,j \in I$ and for $h \in \mathcal{D}(\Theta)$,

$$\int_Y g_i(y)\overline{g_j(y)}/||\theta_y||^2 d\eta(y) = \delta_{ij}$$
$$\int_Y g_i(y)/||\theta_y||\,\langle h,\theta_y/||\theta_y||\rangle d\eta(y) = \int_Y g_i(y)\langle h,\theta_y\rangle d\mu(y) = 0.$$

To complete the proof of (2), we then assume without loss of generality that Θ is normalized and the range of its analysis operator has co-dimension m. We claim Z is a (countable) union of atoms. If not, given $n \in \mathbb{N}$, there exists a disjoint collection

$M_1, M_2, \ldots, M_{m+1}$ of sets of positive measures such that $0 < \mu(\cup_{k=1}^{m+1} M_k) \leq 1/n$. Let $f_k = \chi_{M_k}$; $k = 1,2,\ldots,m+1$. Since $f_1, f_2, \ldots, f_{m+1}$ are mutually orthogonal nonzero vectors, the finite co-dimensional (closed) subspace $\mathcal{R}(T_\Theta)$ contains a linear combination $u_n = \sum_{k=1}^{m+1} \alpha_k f_k$ of L^∞-norm 1. Then $||u_n||^2 = \sum_k |\alpha_k|^2 \mu(M_k) \leq \mu(\cup_k M_k) \to 0$ as $n \to \infty$. Let $h_n \in \mathcal{D}(T_\Theta)$ be such that $T_\Theta h_n = u_n/||u_n||$, $n = 1,2,3,\ldots$. Then $||h_n|| \leq c^{-1}||T_\Theta h_n||_2 = c^{-1}$, where c is the lower bound in the definition of a lower bounded analytic frame. Moreover, for all $n \in \mathbb{N}$,

$n^{1/2} \leq ||u_n||_\infty / ||u_n||_2 = ||T_\Theta h_n||_\infty = sup_z |\langle h_n, \theta_z\rangle|$
$\leq ||h_n||_2 \leq c^{-1}||T_\Theta h_n||_2 = c^{-1}||u_n||_2/||u_n||_2 \leq c^{-1}$;

a contradiction.

Our final result is the construction of examples of pseudo-frames which are lower bounded but not algebraic.

Example 1. Let $\Theta = (\theta_z)_{z\in Z}$ be any generalized frame indexed by a non-discrete measure space $(Z, \mathcal{M}, \mu)$. By Theorem 2, $\mathcal{R}(T_\Theta)$ is an infinite-dimensional closed subspace of $L^2(\mu)$ with infinite co-dimension. Choose arbitrary Hamel bases $\{u_i : i \in I\}$ for $\mathcal{R}(T_\Theta)$ and $\{v_j : j \in J\}$ for $\mathcal{R}(T_\Theta)^\perp$ with $I \subset J$. Define $A: \mathcal{R}(T_\Theta) \to \mathcal{R}(T_\Theta)^\perp$ by $Au_i = v_i$ for all $i \in I$. Next, define the family of linear functionals $\Omega = (\omega_z)_{z\in Z}$ by $\langle h, \omega_z\rangle = (T_\Theta h)(z) + (AT_\Theta h)(z)$, $z \in Z$. Then $t_\Omega h = T_\Theta \oplus AT_\Theta$ which by a suitable choice of A can be nonclosable. Also, for all $h \in H$, $||t_\Omega h||^2 = ||T_\Theta h||^2 + ||AT_\Theta h||^2 \geq ||T_\Theta h||^2 \geq c||h||^2$.

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