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A stochastic perspective to random ship heave motion based on different noises

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Abstract

The stochastic models for ship heave motion in irregular sea waves based on white and colored noises are examined. For this purpose the deterministic model transfers to stochastic models by adding different noise terms in force and then these models are solved analytically and numerically. Finally an investigation is undertaken to examine the parameter estimation problem of second order stochastic differential equations when some of the measurements are unavailable or missing data. Several simulation results are presented to illustrate the performance of the estimators.

Keywords: Colored noise; least square estimation; ship heave motion; stochastic differential equations; white noise

1. Introduction

Stochastic Differential Equations (SDE's) are a natural choice to model a time evolution of dynamic systems which are subject to random influence. For example, in engineering the dynamic of mechanical devices is described by differential equations under the influence of process noise as errors of measurement [1]. Prediction of the behavior of ships in irregular sea remains one of the most difficult problems in ship engineering. Many hydrodynamic studies have included attempts to improve prediction of the motion of a floating structure, and in the wave frequency domain a designer can now calculate the motions with adequate accuracy. The low frequency motion of floating marine structures such as moored ships and oil platforms are one of the main concerns in ocean engineering. Much research has been devoted to obtaining the approximate probability density function (PDF) solutions of nonlinear stochastic oscillators for ship roll motion. For example, [2, 3] used linearization procedure. In the special case of Gaussian white noise excitations, this procedure is equivalent to another method called Gaussian closure [4]. The stochastic average method is another powerful method for the PDF solutions of response amplitude of nonlinear systems [5]. The principle of maximum entropy is another analytical method for approximate PDF solutions [6]. A finite-element method was used to approximate the solutions of Fokker-Plank-Kolmogorov equations [7]. The realistic wave is colored noise, for a special condition the wave excitation is treated as white noise in order to simplify the computations. However, according to our latest information from the research works, the stochastic model for ship heave motion with white and colored noise processes and the parameter estimation of these models have not been studied before.

While the analysis of stochastic processes has been studied extensively over a long period, the estimation of the parameters of second order SDEs have received less attention. Thus in estimating the parameters, both the drift and diffusion terms of these linear and nonlinear models are interesting topics themselves, especially when assessing the relative importance of deterministic and stochastic components of the process under study. In the case of first order diffusion processes driven by Brownian motions, a popular method is the Maximum likelihood estimator (MLE), based on Girsonov density, when the process can be observed continuously [8]. When a diffusion process is observed only at discrete times, in most cases the transition density and hence the likelihood function of the observations, is not explicitly computable. In order to overcome this difficulty, least square estimator (LSE) has been proposed. For the LSE the convergence in probability is proved in [9, 10]. The strong consistency is studied in [11] and the asymptotic distribution was studied in [12]. For more comprehensive discussion, we refer to [13] and the

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references therein.

Estimation of the SDE parameters requires the solution or an approximation to it, then the rest of this paper is organized as follows. Section 2 represents the deterministic and stochastic models for ship heave motion driven by white and colored noise processes and then solves these models analytically. Section 3 describes least square estimator and applies it to finding unknown parameters in ship heave equation. Section 4 performs numerical example and simulates the confidence interval for the expectations of solutions in order to show the accuracy of the present work. Finally, conclusions and future directions will be presented in section 5.

2. Mathematical model

2.1. The deterministic model

The vertical motion of the body in the still water follows from Newton's second law [14],

$$\frac{d}{dt}(\rho v.\dot{z}) = F_h + F_w, \tag{1}$$

in which $\rho =$ density of water, $v =$ volume of displacement of the body, $F_h =$ hydromechanical force in the Z-direction, $F_w =$ exciting wave force in the Z-direction.

Vertical motions of the cylinder are determined by the solid mass of the cylinder and the hydromechanical load on the cylinder. Applying Newton's second law for the heaving cylinder,
$m\ddot{z} = -P + pA_w - b\dot{z} - a\ddot{z} = -P + \rho g(T - z)A_w - b\dot{z} - a\ddot{z}.$

By Archimedes law $P = \rho gTA_w$, the linear equation of the heave motion becomes,

$$\begin{aligned}(m+a)\ddot{z} + b\dot{z} + cz = 0,\\ z(0) = z_a,\ \dot{z}(0) = 0,\end{aligned} \tag{2}$$

in which:

Nomenclature

z	vertical displacement
$P = mg$	mass force downwards
$m = \rho A_w T$	solid mass of cylinder
a	hydrodynamic mass coefficient
b	hydromechani damping coefficient
$c = \rho g A_w$	restoring spring coefficient
$A_w = \frac{\pi}{4}D^2$	water plane area
D	diameter of cylinder
T	draft of cylinder at rest
w_e	circular frequency of encounter
M_3	resulting external force in the Z direction

the terms $a\ddot{z}$ and $b\dot{z}$ are caused by the hydrodynamic reaction as a result of the movement of the cylinder with respect to the water. The water is assumed to be ideal and to behave as in a potential flow.

The solution of the Eq. (2) is as follows

$$z(t) = z_a e^{-\nu t}(\cos w_z t + \frac{\nu}{w_z}\sin w_z t), \tag{3}$$

where $2\nu = \frac{b}{m+a},\ w_0^2 = \frac{c}{m+a},\ w_z^2 = w_0^2 - \nu^2$.

The equation of motion in sea with regular waves are as follows

$$(m+a)\ddot{z} + b\dot{z} + cz = F_t = M_3 cos(w_e t). \tag{4}$$

The solution of Eq. (4) is the form

$$z(t) = e^{-\nu t}(c_1\cos(w_z t) + c_2\sin(w_z t)) + z_a\cos(w_e t + \varepsilon).$$

2.2. The stochastic model with white noise

Modeling of physical systems by ordinary differential equations ignores stochastic effects, by adding random elements into the differential equations, a system of stochastic differential equations arises.

Definition 1. A continuous time random process W(t) is a white noise process if its mean function and autocorrelation function satisfy the following:

(1) $\mu(t) = E[W(t)] = 0,$

(2) $E[W(t_1)W(t_2)] = (\frac{N_0}{2})\delta(t_1 - t_2),$

where $\delta(t)$ is the Dirac delta function.

Now, let us allow some randomness in the force, so force may not be deterministic but it is of the form,

$$F^*(t) = F(t) +' noise' = F(t) + \alpha W(t), \tag{5}$$

where $W(t)$ is a white noise process of mean zero and variance one, and α is nonnegative constant known as the intensity of noise. Then in stochastic form, Eq. (4) can be written as:

$$(m+a)\ddot{z} + b\dot{z} + cz = F_t^* = M_3 cos(w_e t) + \alpha W(t). \tag{6}$$

This is a second order stochastic differential equation. In order to solve this equation analytically, let us consider $X(t) = \begin{pmatrix} z(t) \\ \dot{z}(t)\end{pmatrix} = \begin{pmatrix} x_1(t) \\ x_2(t)\end{pmatrix}$ and obtain

$$\begin{cases} dx_1(t) = x_2(t)dt, \\ (m+a)dx_2(t) = -b\cdot x_2(t)dt - c\cdot x_1(t)dt + M_3\cos(\omega_e t)dt + \alpha W_t dt. \end{cases} \quad (7)$$

$W_t dt = dB_t$ is one dimensional Brownian motion, so Eq. (7) in matrix form is

$$dX(t) = AX(t)dt + H(t)dt + KdB(t), \quad (8)$$

where

$$dX = \begin{pmatrix} dx_1(t) \\ dx_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{-c}{m+a} & \frac{-b}{m+a} \end{pmatrix},$$

$$H(t) = \begin{pmatrix} 0 \\ \frac{M_3\cos(\omega_e t)}{m+a} \end{pmatrix}, K = \begin{pmatrix} 0 \\ \frac{\alpha}{m+a} \end{pmatrix}.$$

We rewrite Eq. (8) as

$$\begin{aligned} &exp(-At)dX(t) - exp(-At)AX(t)dt \\ &= exp(-At)[H(t)dt + KdB(t)], \end{aligned} \quad (9)$$

where for a general $n*n$ matrix F we define $exp(F)$ to be the $n*n$ matrix given by $exp(F) = \sum_{n=0}^{\infty} \frac{1}{n!}F^n$. We use a 2 dimensional version of the Ito formula, applying this results in the two coordinate functions g_1, g_2 of $g:[0,\infty)\times R^2 \to R^2$ given by $g(t,x_1,x_2) = exp(-At)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. We obtain the

$$\begin{aligned} &d(exp(-At)X(t)) \\ &= (-A)exp(-At)X(t)dt + exp(-At)dX(t), \end{aligned}$$

substituted in Eq. (9) this gives,

$$X(t) = e^{At}X(0) + \int_0^t e^{-A(s-t)}H(s)ds + \int_0^t e^{-A(s-t)}KdB(s). \quad (10)$$

The solution $X(t)$ is a random process and its expectation by using theorem 1 is given as

$$E(X(t)) = e^{At}E(X_0) + \int_0^t e^{-A(s-t)}H(s)ds, \quad (11)$$

for every $t > 0$.

Theorem 1. Let $(\Omega, \mathsf{F}, \mathsf{P})$ be a basic probability space equipped with a right continuous family of σ-algebras $\{\mathsf{F}_t, t \ge 0\}$ and f_t is F_t-adapted function, and let $0 \le S < T$. Then

$$E[\int_S^T f(t)dB_t] = 0.$$

Proof 1. See [15].

Lemma 1. *let* $A = \begin{pmatrix} 0 & 1 \\ -a & -b \end{pmatrix}$ then

$$exp(At) = \frac{exp(-\lambda t)}{\xi}\{(\xi\cos(\xi t) + \lambda\sin(\xi t))I + A\sin(\xi t)\},$$

where,

$$\lambda = \frac{b}{2}, \xi = \sqrt{a - \frac{b^2}{4}}.$$

Proof 2. The characteristic function of matrix A is

$$f(x) = |xI - A| = x^2 + bx + a = 0,$$

If γ_1 and γ_2 are the roots of $f(x)$, we have

$$\gamma_1 = \frac{-b}{2} + \sqrt{a - \frac{b^2}{4}}i, \gamma_2 = \frac{-b}{2} - \sqrt{a - \frac{b^2}{4}}i,$$

let $\lambda = \frac{-b}{2}$ and $\xi = \sqrt{a - \frac{b^2}{4}}$, then $\gamma_1 = \lambda + \xi i$ and $\gamma_2 = \lambda - \xi i$. It can be easily shown that,

$$\begin{aligned} e^{At} &= e^{(\lambda+\xi i)}I + \frac{e^{(\lambda+\xi i)t} - e^{(\lambda-\xi i)t}}{2\xi i}[A - (\lambda - \xi i)I] \\ &= e^{\lambda t}\{e^{\xi i t}I + \frac{e^{\xi i t} - e^{-\xi i t}}{2\xi i}[A - \lambda I - \xi i I]\} \\ &= e^{\lambda t}\{(\cos\xi t + i\sin\xi t)I + \frac{\sin\xi t}{\xi}(A - \lambda I - \xi i I)\} \\ &= \frac{e^{\lambda t}}{\xi}\{(\xi\cos\xi t - \lambda\sin\xi t)I + A\sin\xi t\}. \end{aligned}$$

If $z(0) = \dot{z}(0) = 0$, then using this lemma

$$E(z(t)) = \int_0^t \frac{F(s)}{m+a}.\frac{e^{-\lambda(t-s)}}{\xi}.\sin\xi(t-s)ds, \quad (12)$$

where

$$\lambda = \frac{b}{2(m+a)} \; and \; \xi = \sqrt{\frac{c}{m+a} - \frac{b^2}{4(m+a)^2}}.$$

2.3. The stochastic model with colored noise

A white noise process cannot be physically realized but it can be approximated by conventional stochastic processes with wide spectral bands which are commonly known as colored noise processes [16]. The realistic wave is colored noise, for a special condition the wave excitation is treated as white noise in order to simplify the computations.

Definition 2 The process $V(t)$ is called colored noise if it is an Orstein-Uhlenbeck process which satisfies the linear SDE

$$dV(t) = \mu V(t)dt + \sigma dB(t), \tag{13}$$

where μ, σ are constants.

The explicit solution of Eq. (13) is

$$V(t) = e^{\mu t}(V(0) + \sigma \int_0^t e^{-\mu s} dB(s)), \tag{14}$$

so $V(t)$ is a stochastic process with expectation $E(V(t)) = e^{\mu t} E(V(0))$. Now, the noise term in Eq. (6) is considered as a colored noise process. By substituting (13) in (6), we have:

$$dX(t) = AX(t)dt + H(t)dt + KdV(t). \tag{15}$$

The explicit solution of Eq. (15) is

$$X(t) = e^{At} X(0) + \int_0^t e^{-A(s-t)} H(s)ds + \int_0^t e^{-A(s-t)} KdV(s). \tag{16}$$

And hence the expectation of $X(t)$ is

$$E(X(t)) = e^{At} E(X_0) + \int_0^t e^{-A(s-t)} H(s)ds + E(\int_0^t e^{-A(s-t)} KdV(s)), \tag{17}$$

where

$$\begin{aligned} E(\int_0^t e^{-A(s-t)} KdV(s)) &= E(\int_0^t e^{-A(s-t)} K(\mu V(s) + \sigma dB(s))) \\ &= E(\int_0^t e^{-A(s-t)} K\mu V(s)) + E(\int_0^t e^{-A(s-t)} K\sigma dB(s)) \\ &= K\mu(\int_0^t e^{-A(s-t)} E(V(s))ds) + 0 \\ &= K\mu E(V(0)) \int_0^t e^{-A(s-t)} e^{\mu s} ds. \end{aligned}$$

If $V(0) = 0$ then $E(V(0)) = 0$, so the expectations of $X(t)$ with white and colored noise are equal each other.

3. Parameter estimation

There is a great potential for the use of assimilation-based parameter estimation in the ocean sciences. Assume that $X(t)$ is observed at some discrete time instances $\{t_i = ih, i = 1,2,\ldots,n\}$. Generally it is difficult to find a closed form solution for many SDE's, so we focus on finding discrete time approximations to the solution of Eq. (7), namely the strong Euler-Maruyama scheme that attains convergence of order 0.5 [17].

The multidimensional Euler method for the Ito system (7) is:

$$\begin{cases} x_1(t_i) - x_1(t_{i-1}) = x_2(t_{i-1}) \cdot \Delta t_{i-1} \\ x_2(t_i) - x_2(t_{i-1}) = (\frac{-b}{m+a} x_2(t_{i-1}) - \frac{c}{m+a} x_1(t_{i-1}) + \frac{F(t_{i-1})}{m+a})\Delta t_{i-1} \\ \qquad + \frac{\alpha}{m+a}(B_{t_i} - B_{t_{i-1}}), \end{cases} \tag{18}$$

where $\Delta t_{i-1} = t_i - t_{i-1}$, for $i = 1,\ldots,n$ and the noise increments $\Delta B_{t_{i-1}} = B_{t_i} - B_{t_{i-1}} \approx N(0, \Delta t_{i-1})$. It is important to identify the parameters a, b and c from the observed data when some of them are unavailable or missing data. This leads us to the mathematical problem of estimating the true values of (a,b,c) from the observations $\{X_{t_i}, i = 1,2,3,\ldots,n\}$.

Since the diffusion process is observed at discrete times, the transition density and hence the likelihood function of observations is not explicitly computable, therefore the least square estimator is used. First, the parameters estimation with white noise is examined.

Let us consider parameter a as unknown, thus the LSE for Eq. (7) is to minimize the following contrast function,

$$\Phi_n(a) = \sum_{i=1}^n |\varphi_{n_i}|^2, \tag{19}$$

where

$$\varphi_{n_i} = (m+a)(x_2(t_i) - x_2(t_{i-1})) + (bx_2(t_{i-1}) + cx_1(t_{i-1}))\Delta t_{i-1} - F(t_{i-1})\Delta t_{i-1}.$$

The LSE $\hat{a}_n$ is defined as $\hat{a}_n = argmin_a \Phi_n(a)$, which can be explicitly represented as,

$$\hat{a}_n = \frac{\sum_{i=1}^n \Psi_{1,i}}{\sum_{i=1}^n \Psi_{2,i}}, \tag{20}$$

where

$$\Psi_{1,i} = [-m(x_2(t_i) - x_2(t_{i-1})) - (bx_2(t_{i-1}) + cx_1(t_{i-1}))\Delta t_{i-1} - F(t_{i-1})\Delta t_{i-1}].[x_2(t_i) - x_2(t_{i-1})],$$

and

$$\Psi_{2,i} = [x_2(t_i) - x_2(t_{i-1})]^2.$$

Now consider Eq. (7) with three unknown parameters a, b and c. We are interested in the estimation of (a,b,c) based on discrete observations $\{X_{t_i}\}$. The LSE of (a,b,c) is obtained by minimizing the contrast function which now becomes

$$\Phi_n(a,b,c) = \sum_{i=1}^{n} |\phi_{n_i}(a,b,c)|^2, \quad (21)$$

where

$$\phi_{n_i}(a,b,c) = (m+a)(x_2(t_i) - x_2(t_{i-1})) + (bx_2(t_{i-1}) + cx_1(t_{i-1}))\Delta t_{i-1} - F(t_{i-1})\Delta t_{i-1}.$$

The LSE $(\hat{a}_n, \hat{b}_n, \hat{c}_n)$ is represented as the solution of the system of equation shown below,

$$\begin{cases} \sum_{i=0}^{n}(x_2(t_i) - x_2(t_{i-1}))[(m+a)(x_2(t_i) - x_2(t_{i-1})) + \\ bx_2(t_{i-1})\Delta t_{i-1} + cx_1(t_{i-1})\Delta t_{i-1} - F(t_{i-1})\Delta t_{i-1}] = 0 \\ \sum_{i=0}^{n}(x_2(t_{i-1})\Delta t_{i-1})[(m+a)(x_2(t_i) - x_2(t_{i-1})) + \\ bx_2(t_{i-1})\Delta t_{i-1} + cx_1(t_{i-1})\Delta t_{i-1} - F(t_{i-1})\Delta t_{i-1}] = 0 \\ \sum_{i=0}^{n}(x_1(t_{i-1})\Delta t_{i-1}))[(m+a)(x_2(t_i) - x_2(t_{i-1})) + \\ bx_2(t_{i-1})\Delta t_{i-1} + cx_1(t_{i-1})\Delta t_{i-1} - F(t_{i-1})\Delta t_{i-1}] = 0. \end{cases} \quad (22)$$

Let us establish the estimation of parameters with colored noiose. In this case the multidimensional Euler method for Eq. (15) is:

$$\begin{cases} x_1(t_i) - x_1(t_{i-1}) = x_2(t_{i-1}) \cdot \Delta t_{i-1} \\ x_2(t_i) - x_2(t_{i-1}) = (\frac{-b}{m+a} x_2(t_{i-1}) - \frac{c}{m+a} x_1(t_{i-1}) + \frac{F(t_{i-1})}{m+a})\Delta t_{i-1} \\ \quad + \frac{\alpha}{m+a}(\mu V(t_{i-1})\Delta t_{i-1} + \sigma(B_{t_i} - B_{t_{i-1}})), \end{cases} \quad (23)$$

The LSE for Eq. (23) is to minimize the following contrast function,

$$\Omega_n(a,b,c) = \sum_{i=1}^{n} |\Omega_{n_i}(a,b,c)|^2, \quad (24)$$

where

$$\Omega_{n_i}(a,b,c) = (m+a)(x_2(t_i) - x_2(t_{i-1})) + (bx_2(t_{i-1}) + cx_1(t_{i-1}))\Delta t_{i-1} - F(t_{i-1})\Delta t_{i-1} - \alpha(\mu V(t_{i-1})\Delta t_{i-1}),$$

then the LSE $\hat{a}_n$, $\hat{b}_n$, $\hat{c}_n$ are defined as,

$$(\hat{a}_n, \hat{b}_n, \hat{c}_n) = argmin_{a,b,c}\Omega_n(a,b,c),$$

4. Numerical simulation

We have applied our estimator to the process determined by the following stochastic differential equation $(m+a)\ddot{z} + b\dot{z} + cz = F(t) + \alpha W_t$. In simulation, let us consider

$$\begin{aligned} &m = 100(kg), \\ &a = 5, b = 3, c = 1, \\ &z_0 = 0, \dot{z}_0 = 0, \\ &F(t) = 200\cos(\frac{\pi}{4}t). \end{aligned} \quad (25)$$

and the used values of n and α vary. Using the Euler-Maruyama method, Z(t) is simulated in intervals [0,1]. Fig. 1 presents the expectation and 95% confidence interval of stochastic solution for $z(t)$ with white and colored noise processes. Numerical simulations for parameters estimation are shown in Table 1, Table 2 and Table 3.

It can be claimed that when $\alpha \to 0$, the obtained estimation values are very close to the exact parameter values.

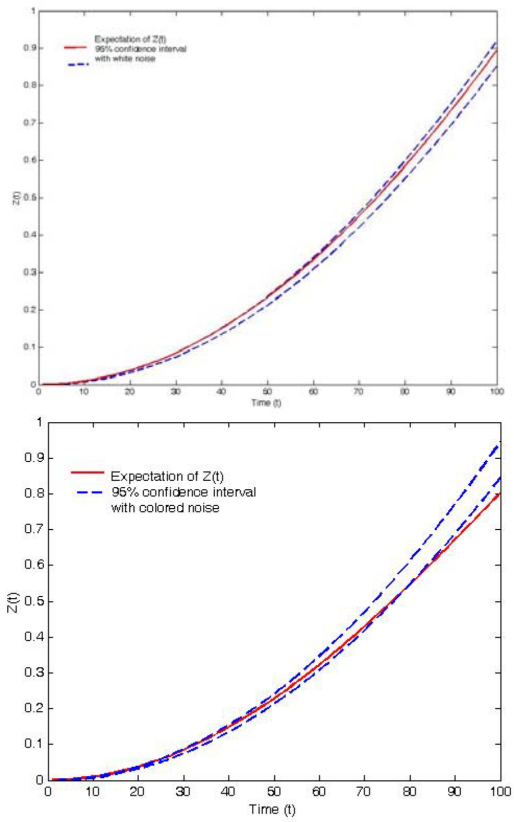

Fig. 1. The expectation and 95% confidence interval of z(t) in interval [0,1] with white and colored noise processes

Table 1. The numerical values of the estimator $\hat{a}$ with 1000 trials

Intensity of noise	Noise terms	n=100	n=200	n=300	n=400	n=500
$\alpha = 1$	white noise	5.0555	5.0225	4.9942	5.0022	4.9908
	colored noise	4.9979	5.0015	4.9997	4.9985	4.9967
$\alpha = 0.1$	white noise	4.9990	5.0018	4.9993	4.9968	5.0000
	colored noise	4.9997	4.9999	4.9998	5.0004	5.0000
$\alpha = 0.01$	white noise	4.9998	5.0001	5.0001	4.9998	5.0000
	colored noise	5.0000	5.0000	5.0000	5.0000	5.0000

Table 2. The numerical values of the estimator $\hat{a}$ with 5000 trials

Intensity of noise	Noise terms	n=100	n=200	n=300	n=400	n=500
$\alpha = 1$	white noise	5.0025	5.0013	4.9975	4.9964	4.9916
	colored noise	5.0008	5.0008	4.9994	4.9995	5.0001
$\alpha = 0.1$	white noise	5.0002	4.9998	4.9995	4.9982	4.9997
	colored noise	5.0000	5.0000	5.0000	5.0000	5.0000
$\alpha = 0.01$	white noise	5.0000	5.0000	5.0000	5.0000	5.0000
	colored noise	5.0000	5.0000	5.0000	5.0000	5.0000

Table 3. The numerical values of the estimators $(\hat{a}_n, \hat{b}_n, \hat{c}_n)$ with 1000 trials

Intensity of noise	Noise terms	Estimators	n=100	n=200	n=300	n=400	n=500
$\alpha = 0.1$	white noise	$\hat{a}_n$	5.0013	4.9995	5.0001	5.0005	5.0001
		$\hat{b}_n$	3.0031	3.0001	2.9945	3.0006	3.0021
		$\hat{c}_n$	0.9915	1.0007	1.0078	0.9988	0.9973
	colored noise	$\hat{a}_n$	5.0009	4.9998	5.0002	5.0001	5.0001
		$\hat{b}_n$	3.001	3.0005	2.9993	3.0003	3.0002
		$\hat{c}_n$	0.9945	0.9991	1.0007	0.9989	1.0001
$\alpha = 0.01$	white noise	$\hat{a}_n$	5.0001	5.0000	5.0000	5.0000	5.0000
		$\hat{b}_n$	2.9996	2.9999	3.0004	2.9999	3.0000
		$\hat{c}_n$	1.0006	1.0000	0.9994	1.0002	1.0001
	colored noise	$\hat{a}_n$	5.0002	5.0000	5.0000	5.0000	5.0000
		$\hat{b}_n$	3.001	3.0005	2.9998	3.0001	3.0000
		$\hat{c}_n$	1.0003	0.9995	1.0002	1.0001	1.0000

5. Conclusion

This paper provides the ship heave motion equation when the force is not deterministic and affected by random elements including white and colored noise processes. Thus some SDE's for modeling ship heave motion in irregular waves are established. These models have been solved analytically and expectations of solutions have been derived. Since the parameters of model can be unknown, LSE is provided to estimate these parameters. In simulation study, we consider that a, b, and c (the parameters of model) have the exact values and then observe $\{X_{t_i}\}_{i=1}^{n}$ by using EM method. With these observations, we estimate $(\hat{a}_n, \hat{b}_n, \hat{c}_n)$ with proposed estimators. Our results show that the estimated values dramatically tend to exact values when $\alpha \rightarrow 0$, and $n \rightarrow \infty$.

We expect to obtain some new and interesting results in proving the convergence of LSE in 2-dimensional SDE's. All these extensions will be in the scope of our ongoing and future projects.

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