IJST (2013) 37A3: 199-210
Iranian Journal of Science & Technology

http://ijsts.shirazu.ac.ir

Fractional lie series and transforms as canonical mappings

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Abstract

Using the Riemann-Liouville fractional differintegral operator, the Lie theory is reformulated. The fractional Poisson bracket over the fractional phase space as 3N state vector is defined to be the fractional Lie derivative. Its properties are outlined and proved. A theorem for the canonicity of the transformation using the exponential operator is proved. The conservation of its generator is proved in a corollary. A Theorem for the inverse fractional canonical mapping is proved. The composite mappings of two successive transformations is defined. The fractional Lie operator and its properties are introduced. Some useful lemmas on this operator are proved. Lie transform depending on a parameter over the fractional phase space is presented and its relations are proved. Two theorems that proved the transformation $\Phi = E_W \mathbf{Z}$ is completely canonical and is a solution of the Hamiltonian system (30) are given. Recurrence relations are obtained.

Keywords: Lie series and transform; fractional Poisson bracket; Hamiltonian systems

1. Introduction

In 1966, the Poincaré and von Zeipel-Brouwer theories were rejuvenated by Hori [1] through the introduction of canonical transformations expressed by Lie series mappings instead of the classical Jacobian transformations.

Over the years many different techniques have been developed for handling various perturbation problems. Some are suited for a few special problems while others are quite general, but almost all were developed before the computer age. To our knowledge only one general technique was developed specifically to be used in conjunction with a computer algebra system, namely the method of Lie transforms. It is truly an algorithm in the sense of modern computer science: a clearly defined iterative procedure. The method was first given in Deprit [2] for Hamiltonian systems of differential equations, then generalized to arbitrary systems of differential equations by Kamel [3] and Henrard [4]. The predecessor of this method was a limited set of formulas given in Hori [1]. All these papers appeared in astronomy journals far from the usual journals of perturbation analysis. Through the seventies only a few papers on this subject appeared outside the astronomy literature. Recently, several books have presented the method but only in the limited context in which it was initially developed.

Received: 22 November 2011 / Accepted: 1 January 2012

In celestial mechanics, the Hamiltonian is usually a periodic or almost periodic function of time, a further requirement would be the averaging of the Hamiltonian to eliminate the time. The canonical form of differential equations offers a possibility of establishing general rules governing transformations from one set of variables to another set. Under these rules the canonical form of the equations is preserved. When choosing the suitable transformation, the original problem may be changed to a simpler one. Often the number of degrees of freedom is reduced, and in some cases the complete solution may thus be achieved.

A theorem by Lie has been applied to construct explicit transformations. Hori [5] constructed an algorithm using the Lie series to determine the transformed Hamiltonian from the old one. Deprit [2] constructed another algorithm to generate the new Hamiltonian recursively using the Lie transform. Cambell and Jefferys [6] and Mersman [7] showed the equivalence of Hori's and Deprit's methods while Kamel [3] simplified Deprit's algorithm. Hori [8] showed that second order, Lie transforms are equivalent to von Zeiple's technique. Shniad [9] proved that the von Zeipel transformation is equivalent to the Deprit's transformation, while Mersman [10] established the equivalence of Hori, Deprit and von Zeipel transformation. Sessin [11] showed that the equations generated by Lie-Deprit's method could be solved in the same way as Hori did in his method. Varadi [12] constructed transformation depending on two parameters. Ahmed [13] developed a method to construct a Lie transform in the multiple-case parameters. Cui and Garfinkel [14] modified Hori's method to achieve the solution, when the unperturbed part of the Hamiltonian depends only on the momenta, without using pseudo-time.

The Fractional Calculus (**FC**) generalizes the ordinary differentiation and integration so as to include any arbitrary irrational order instead of being only the positive integers (see Samko et. al. [15], Kilbas, et. al. [16], Magin [17], Podlubny [18]). In a letter to L'Hopital in 1965, Leibniz raised the possibility of generalizing the operation of differentiation to non-integer orders, and L'Hopital asked what would be the result of half-differentiating x. Leibniz replied, It leads to a paradox, from which one day useful consequences will be drawn. The paradoxical aspects are due to the fact that there are several different ways of generalizing the differentiation operator to non-integer powers, leading to inequivalent results. During the second half of the twentieth century, many authors have explored the world of **FC** giving new insight into many areas of scientific research in physics, mechanics and mathematics. Miller and Ross [19] pointed out, there is hardly a field of science or engineering that has remained untouched by the new concepts of **FC**. To move from the integer-order calculus to the **FC** version of a system we replace the time derivative in an evolution equation with a derivative of fractional order. Riewe [20], [21] has formulated Lagrangian and Hamiltonian mechanics to include derivatives of fractional order. It has been shown that Lagrangian involving fractional time derivatives leads to equations of motion with non conservative classical forces such as friction using certain functionals. In these references, fractional derivative terms were introduced in functionals to obtain nonconservative terms in the desired differential equation. Agrawal in a very interesting series of papers, see Agrawall [22-24] has developed fractional calculus of variations dealing with Lagrangian involving Riemann-Liouville (R-L) fractional derivatives. He has presented fractional Euler-Lagrange equations involving Caputo derivatives. Baleanu and Muslih [25], [26] developed a fractional Hamiltonian in terms of Caputo derivatives. Baleanu [27] compared the results of fractional Euler-Lagrange equations corresponding to several fractional generalized derivatives. He presented fractional Lagrangians which differ by a fractional Riesz derivative. He showed the difference of the obtained fractional Euler-Lagrange equations when several fractional derivatives are used, namely the Riemann-Liouville, Caputo and Riesz derivatives.

In fact, Poisson brackets constitute an important part of Hamiltonian mechanics. Therefore a generalization of Poisson-bracket (fractional version) is introduced. Using this fractional Poisson-bracket, some very useful theorems and Lemmas that are required for Lie series and transform are proved. A final word is with the fractional mechanics. In a forthcoming work we hope to prove both conservative and non-conservative systems with only one equation of motion.

2. Riemann-Liouville Fractional Operator

The development of the **FC** theory is due to the contributions of many mathematicians such as Euler, Liouville, Riemann, and Letnikov. Several definitions of a fractional derivative have been proposed. These definitions include Riemann-Liouville, Grunwald-Letnikov,Weyl, Caputo, Marchaud, and Riesz fractional derivatives, see Miller and Ross [19], Riewe [20], and Baleanu [27]. Riemann-Liouville derivative is the most used generalization of the derivative. It is based on the direct generalization of Cauchy's formula for calculating an n-fold or repeated integral. The right and the left Riemann-Liouville fractional derivatives, in brief are denoted by **RRLFD** and **LRLFD** respectively, [28] can be written as,

$$ {}_aD_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \times \int_a^x f(t)(x-t)^{n-\alpha-1}\,dt \quad (1)$$

$$ {}_xD_b^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^n \times \int_x^b f(t)(x-t)^{n-\alpha-1}\,dt \quad (2)$$

where Γ represents the Euler gamma function, α is the order of the derivative such that $n-1 \le \alpha < n$. If α is an integer, these derivatives are defined in the usual sense,

$$\text{i.e } {}_aD_x^\alpha f(x) = \left(\frac{d}{dx}\right)^\alpha f(x),$$

and

$$ {}_xD_b^\alpha f(x) = \left(-\frac{d}{dx}\right)^\alpha f(x)$$

The fractional operator ${}_aD_x^\alpha$ can be written as [29],

$$ {}_aD_x^\alpha = \left(\frac{d^n}{dx^n}\right) {}_aD_x^{\alpha-n} \quad (3)$$

where the number of additional differentiations n is equal to $[\alpha]+1$, where $[\alpha]$ is the whole part of α. The operator ${}_aD_x^\alpha$ is a generalization of differential and integral operators and can be introduced as follows:

$$ {}_aD_x^\alpha = \begin{cases} \dfrac{d^\alpha}{dx^\alpha}, & \text{Fractional differentiation mapping} \\ 1, & \text{Fractional unitary mapping} \\ \int_a^x dt^{-\alpha} & \text{Fractional integration mapping} \end{cases}$$

Definition 1. Let $\mathcal{A}_{(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta)} \in \mathbb{R}^{3n}$ be the set of all real analytic functions of $3n$ canonically conjugated variables $\mathbf{Z} = (\mathbf{u}, \mathbf{U}_\alpha, \mathbf{U}_\beta)$ which define the fractional phase space as $3N$ state vector, and let $\mathcal{F}, W \in \mathcal{A}_{(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta)}$. The Fractional Lie derivative $\mathfrak{L}_W$ generated by W over the fractional phase space is defined by the generalized Poission bracket $\mathfrak{L}_W = (\mathcal{F};W)$ [30]

$$\begin{aligned} \mathfrak{L}_W \mathcal{F} = & \sum_{j=1}^{N} \left\{ \frac{\partial \mathcal{F}}{\partial u_j} \left(\frac{\partial W}{\partial U_{\alpha j}} + \frac{\partial W}{\partial U_{\beta j}} \right) \right. \\ & \left. - \frac{\partial W}{\partial u_j} \left(\frac{\partial \mathcal{F}}{\partial U_{\alpha j}} + \frac{\partial \mathcal{F}}{\partial U_{\beta j}} \right) \right\} \end{aligned} \quad (4)$$

where $\mathbf{U}_\alpha$ and $\mathbf{U}_\beta$ the fractional canonical momenta are obtained by replacing the ordinary differentiation of the generalized coordinates $\mathbf{u}$ with respect to time by the fractional differentiation as

$$\mathbf{U}_\alpha = \frac{\partial \mathcal{L}\left(\mathbf{u}, {}_aD_t^\alpha \mathbf{u}, {}_tD_b^\beta \mathbf{u}, \mathbf{t}\right)}{\partial\, {}_aD_t^\alpha \mathbf{u}},$$

$$\mathbf{U}_\beta = \frac{\partial \mathcal{L}\left(\mathbf{u}, {}_aD_t^\alpha \mathbf{u}, {}_tD_b^\beta \mathbf{u}, \mathbf{t}\right)}{\partial\, {}_tD_b^\beta \mathbf{u}}$$

note that $\mathbf{U}_\alpha$ and $\mathbf{U}_\beta$ are independent, and $\mathcal{L}$ is the Lagrangian of the problem.

2.1 Properties of the Fractional Lie derivative

Let $\mathcal{F}, \mathcal{K}, W \in \mathcal{A}_{(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta)}$ and $\alpha, \beta \in \mathbb{R}$, $\mathbb{R}$ be the real numbers, then the following properties hold

$$\left.\begin{aligned} \mathfrak{L}_W(\alpha\mathcal{F} + \beta\mathcal{K}) &= \alpha\mathfrak{L}_W(\mathcal{F}) + \beta\mathfrak{L}_W(\mathcal{K}) \\ \mathfrak{L}_W(\mathcal{F}\cdot\mathcal{K}) &= \mathcal{K}\cdot\mathfrak{L}_W\mathcal{F} + \mathcal{F}\cdot\mathfrak{L}_W\mathcal{K} \\ \mathfrak{L}_W(\mathcal{F};\mathcal{K}) &= (\mathcal{F};\mathfrak{L}_W\mathcal{K}) + (\mathfrak{L}_W\mathcal{F};\mathcal{K}) \\ \mathfrak{L}_V\mathfrak{L}_W &= \mathfrak{L}_W\mathfrak{L}_V + \mathfrak{L}_{(W;V)} \end{aligned}\right\} \quad (5)$$

The first two properties of (5) can be proved directly and the last two properties of (5) can be proved as follows:

Proof: The third property can be proved as;

$$\begin{aligned} &\sum_{j=1}^{N}\left\{ \frac{\partial(\mathcal{F};\mathcal{K})}{\partial u_j}\left(\frac{\partial W}{\partial U_{\alpha j}} + \frac{\partial W}{\partial U_{\beta j}}\right) - \frac{\partial W}{\partial u_j}\left(\frac{\partial(\mathcal{F};\mathcal{K})}{\partial U_{\alpha j}} + \frac{\partial(\mathcal{F};\mathcal{K})}{\partial U_{\beta j}}\right)\right\} + \sum_{j=1}^{N}\left\{\frac{\partial(W;\mathcal{F})}{\partial u_j}\left(\frac{\partial\mathcal{K}}{\partial U_{\alpha j}} + \frac{\partial\mathcal{K}}{\partial U_{\beta j}}\right)\right. \\ &\left. - \frac{\partial\mathcal{K}}{\partial u_j}\left(\frac{\partial(W;\mathcal{F})}{\partial U_{\alpha j}} + \frac{\partial(W;\mathcal{F})}{\partial U_{\beta j}}\right)\right\} + \sum_{j=1}^{N}\left\{\frac{\partial(\mathcal{K};W)}{\partial u_j}\right. \\ &\left. \times\left(\frac{\partial\mathcal{F}}{\partial U_{\alpha j}} + \frac{\partial\mathcal{F}}{\partial U_{\beta j}}\right) - \frac{\partial\mathcal{F}}{\partial u_j}\left(\frac{\partial(\mathcal{K};W)}{\partial U_{\alpha j}} + \frac{\partial(\mathcal{K};W)}{\partial U_{\beta j}}\right)\right\} = 0 \end{aligned}$$

After some lengthy algebraic manipulation we can verify that the Jacobi identity holds true in case of using the fractional operator

$$((\mathcal{F};\mathcal{K});W) + ((W;\mathcal{F});\mathcal{K}) + ((\mathcal{K};W);\mathcal{F}) = 0$$
$$((\mathcal{F};\mathcal{K});W) = ((\mathcal{F};W);\mathcal{K}) + (\mathcal{F};(\mathcal{K};W))$$
$$\mathfrak{L}_W(\mathcal{F};\mathcal{K}) = (\mathfrak{L}_W\mathcal{F};\mathcal{K}) + (\mathcal{F};\mathfrak{L}_W\mathcal{K})$$

Proof: The fourth property can be proved as:

$$\mathfrak{L}_V\mathfrak{L}_W\mathcal{F} = \mathfrak{L}_V(\mathcal{F};W) = ((\mathcal{F};W);V)$$

Using the Jacobi identity we obtain

$$((\mathcal{F};W);V) + ((V;\mathcal{F});W) + ((W;V);\mathcal{F}) = 0$$
$$((\mathcal{F};W);V) = ((\mathcal{F};V);W) + (\mathcal{F};(W;V))$$
$$\mathfrak{L}_V(\mathcal{F};W) = \mathfrak{L}_W(\mathcal{F};V) + \mathfrak{L}_{(W;V)}\mathcal{F}$$

The n th iterate of the Lie derivative is defined as follows

$$\mathfrak{L}_W^0\mathcal{F} = \mathcal{F},\ \mathfrak{L}_W^1\mathcal{F} = \mathfrak{L}_W^1\mathcal{F},\ \mathfrak{L}_W^n\mathcal{F} = \mathfrak{L}_W\mathfrak{L}_W^{n-1}\mathcal{F}$$

and the corresponding first three properties are

$$\left.\begin{aligned} &\mathfrak{L}_W^n\left(\alpha\mathcal{F}+\beta\mathcal{K}\right)=\alpha\mathfrak{L}_W^n\left(\mathcal{F}\right)+\beta\mathfrak{L}_W^n\left(\mathcal{K}\right)\\ &\mathfrak{L}_W^n\left(\mathcal{F}\cdot\mathcal{K}\right)=\sum_{m=0}^{n}C_m^n\mathfrak{L}_W^m\mathcal{F}\cdot\mathfrak{L}_W^{n-m}\mathcal{K}\\ &\mathfrak{L}_W^n\left(\mathcal{F};\mathcal{K}\right)=\sum_{m=0}^{n}C_m^n\left(\mathfrak{L}_W^m\mathcal{F};\mathfrak{L}_W^{n-m}\mathcal{K}\right)\end{aligned}\right\}\qquad(6)$$

2.2. The Exponential Operator

Definition 2. Since $W\in\mathcal{A}_{\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta\right)}$ is real analytic, we may choose ε sufficiently small such that the exponential operator $\exp\left(\varepsilon\mathfrak{L}_W\right)$ exists at all points $\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta\right)$ of the fractional phase space as an operator function analytic at $\varepsilon=0$. Now we can define the exponential operator as

$$\exp\left(\varepsilon\mathfrak{L}_W\right)=\sum_{n\geq 0}\frac{\varepsilon^n}{n!}\mathfrak{L}_W^n \qquad (7)$$

The formulas corresponding to (6) are

$$\left.\begin{aligned} &\exp\left(\varepsilon\mathfrak{L}_W\right)\left(\alpha\mathcal{F}+\beta\mathcal{K}\right)=\alpha\exp\left(\varepsilon\mathfrak{L}_W\right)\mathcal{F}+\beta\exp\left(\varepsilon\mathfrak{L}_W\right)\mathcal{K}\\ &\exp\left(\varepsilon\mathfrak{L}_W\right)\left(\mathcal{F}\cdot\mathcal{K}\right)=\exp\left(\varepsilon\mathfrak{L}_W\right)\mathcal{F}\cdot\exp\left(\varepsilon\mathfrak{L}_W\right)\mathcal{K}\\ &\exp\left(\varepsilon\mathfrak{L}_W\right)\left(\mathcal{F};\mathcal{K}\right)=\left(\exp\left(\varepsilon\mathfrak{L}_W\right)\mathcal{F};\exp\left(\varepsilon\mathfrak{L}_W\right)\mathcal{K}\right)\end{aligned}\right\}(8)$$

Theorem 3. The transformation $\mathbf{Z}\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta\right)\rightarrow\boldsymbol{\xi}\left(\mathbf{u}^*,\mathbf{U}_\alpha^*,\mathbf{U}_\beta^*\right)$ given by $\boldsymbol{\xi}=\exp\left(\varepsilon\mathfrak{L}_W\right)\mathbf{Z}\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta\right)$ is completely canonical

Proof:

$$\begin{aligned}&\exp\left(\varepsilon\mathfrak{L}_W\right)\left(\boldsymbol{\xi}_i\ ;\ \boldsymbol{\xi}_j\right)\\ &\quad=\left(\exp\left(\varepsilon\mathfrak{L}_W\right)\mathbf{Z}_i\ ;\ \exp\left(\varepsilon\mathfrak{L}_W\right)\mathbf{Z}_j\right)\\ &\quad=\left(\exp\left(\varepsilon\mathfrak{L}_W\right)\right)\left(\mathbf{Z}_i,\mathbf{Z}_j\right)\\ &\quad=\left(1+\varepsilon\mathfrak{L}_W+\frac{\varepsilon^2}{2}\mathfrak{L}_W^2+....\right)\\ &\quad\times\left(\frac{\partial\mathbf{Z}_i}{\partial\mathbf{Z}^\mu}\right)^T J^{\mu\nu}\left(\frac{\partial\mathbf{Z}_j}{\partial\mathbf{Z}^\nu}\right)\\ &\quad=\left(1+\varepsilon\mathfrak{L}_W+\frac{\varepsilon^2}{2}\mathfrak{L}_W^2+....\right)\\ &\quad\times\delta_\mu^i J^{\mu\nu}\delta_\nu^j=J^{ij}\end{aligned}$$

where $J^{\mu\nu}=\begin{bmatrix}I & 0\\ 0 & -I\end{bmatrix}$ is the usual $2n\times 2n$ skew symmetric simplectic identity matrix, and hence the ξ is a canonical set. This is of course because $\exp\left(\varepsilon\mathfrak{L}_W\right)$ acting on the constant matrix $J^{\mu\nu}$ just leaves it invariant.

Remark 4. An intermediate result is that

$$\boldsymbol{\xi}_i=\exp\left(\varepsilon\mathfrak{L}_W\right)\mathbf{Z}_i$$

$$\boldsymbol{\xi}_i=\mathbf{Z}_i+\sum_{n=1}^{\infty}\frac{\varepsilon^n}{n!}\mathfrak{L}_W^{n-1}\mathfrak{L}_W^1\ \mathbf{Z}_i$$

$$\boldsymbol{\xi}_i=\mathbf{Z}_i+\sum_{n=1}^{\infty}\frac{\varepsilon^n}{n!}\mathfrak{L}_W^{n-1}\sum_{i=1}^{N}\left\{\frac{\partial\mathbf{Z}_i}{\partial u_i}\left(\frac{\partial W}{\partial U_{\alpha i}}+\frac{\partial W}{\partial U_{\beta i}}\right)\right.$$

$$\left.-\frac{\partial W}{\partial u_i}\left(\frac{\partial\mathbf{Z}_i}{\partial U_{\alpha i}}+\frac{\partial\mathbf{Z}_i}{\partial U_{\beta i}}\right)\right\}\qquad(9)$$

in particular

$$\left.\begin{aligned}&u_j^* = u_j+\sum_{n=1}^{\infty}\frac{\varepsilon^n}{n!}\mathfrak{L}_W^{n-1}\left(\frac{\partial W}{\partial U_{\alpha i}}+\frac{\partial W}{\partial U_{\beta i}}\right)\\ &U_{\alpha j}^*+U_{\beta j}^*=U_{\alpha j}+U_{\beta j}-\sum_{n=1}^{\infty}\frac{\varepsilon^n}{n!}\mathfrak{L}_W^{n-1}\frac{\partial W}{\partial u_j}\end{aligned}\right\}\qquad(10)$$

Theorem 5. The image of a real analytic function $\mathcal{F}\left(\boldsymbol{\xi}\right)$ under the mapping $\boldsymbol{\xi}=\exp\left(\varepsilon\mathfrak{L}_W\right)\mathbf{Z}$ is $\tilde{\mathcal{F}}\left(\mathbf{Z},\boldsymbol{\varepsilon}\right)=\mathcal{F}\left(\exp\left(\varepsilon\mathfrak{L}_W\right)\mathbf{Z}\right)=\exp\left(\varepsilon\mathfrak{L}_W\right)\mathcal{F}\left(\mathbf{Z}\right)$

Proof:

$$\tilde{\mathcal{F}}\left(\mathbf{Z},\boldsymbol{\varepsilon}\right)=\sum_{n=1}^{\infty}\frac{\varepsilon^n}{n!}\frac{\partial^n\tilde{\mathcal{F}}}{\partial\varepsilon^n}\bigg|_{\varepsilon=0}\qquad(11)$$

but

$$\frac{\partial\tilde{\mathcal{F}}}{\partial\varepsilon}=\frac{\partial\mathcal{F}\left(\boldsymbol{\xi}\right)}{\partial\boldsymbol{\xi}^\mu}\frac{\partial\boldsymbol{\xi}^\mu}{\partial\varepsilon}=\frac{\partial\mathcal{F}\left(\boldsymbol{\xi}\right)}{\partial\boldsymbol{\xi}^\mu}\mathfrak{L}_W\ \boldsymbol{\xi}^\mu,$$

$$\mu=1,2,3,,...3n$$

$$\frac{\partial\tilde{\mathcal{F}}}{\partial\varepsilon}=\frac{\partial\mathcal{F}\left(\boldsymbol{\xi}\right)}{\partial\boldsymbol{\xi}^\mu}\left(\frac{\partial\boldsymbol{\xi}^\mu}{\partial\mathbf{Z}^\lambda}\right)^T J^{\lambda\sigma}\left(\frac{\partial W}{\partial\mathbf{Z}^\sigma}\right)$$

$$\frac{\partial\tilde{\mathcal{F}}}{\partial\varepsilon}=J^{\lambda\sigma}\frac{\partial\mathcal{F}\left(\boldsymbol{\xi}\right)}{\partial\boldsymbol{\xi}^\mu}\left(\frac{\partial\boldsymbol{\xi}^\mu}{\partial\mathbf{Z}^\lambda}\right)^T\left(\frac{\partial W}{\partial\mathbf{Z}^\sigma}\right)$$

$$\frac{\partial \tilde{\mathcal{F}}}{\partial \varepsilon} = \left(\frac{\partial \tilde{\mathcal{F}}}{\partial \mathbf{Z}^{\lambda}}\right)^T J^{\lambda\sigma}\left(\frac{\partial W}{\partial \mathbf{Z}^{\sigma}}\right) = \mathfrak{L}_W \ \tilde{\mathcal{F}} \quad (12)$$

and the n^{th} - iterates give

$$\mathfrak{L}_W^n \tilde{\mathcal{F}} = \frac{\partial^n \tilde{\mathcal{F}}}{\partial \varepsilon^n} \quad (13)$$

and

$$\left.\frac{\partial^n \tilde{\mathcal{F}}}{\partial \varepsilon^n}\right|_{\varepsilon=0} = \mathfrak{L}_W^n \mathcal{F}(\mathbf{Z},0) = \mathfrak{L}_W^n \mathcal{F}(\mathbf{Z})$$

so that (11) yields

$$\tilde{\mathcal{F}}(\mathbf{Z},\varepsilon) = \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!} \mathfrak{L}_W^n \mathcal{F}(\mathbf{Z})$$

or

$$\mathcal{F}\left(\exp(\varepsilon \mathfrak{L}_W)\mathbf{Z}\right) = \exp(\varepsilon \mathfrak{L}_W)\mathcal{F}(\mathbf{Z}) \quad (14)$$

Corollary 6. The generator W is conserved under the canonical mapping $\boldsymbol{\xi} = \exp(\varepsilon \mathfrak{L}_W)\mathbf{Z}$ such that

$$W(\boldsymbol{\xi}) = W\left(\exp\left(\varepsilon \sum_{j=1}^{N}\left\{\frac{\partial}{\partial u_j}\left(\frac{\partial W}{\partial U_{\alpha j}} + \frac{\partial W}{\partial_t D_b^{\beta} u_j}\right) - \frac{\partial W}{\partial u_j}\left(\frac{\partial}{\partial U_{\alpha j}} + \frac{\partial}{\partial U_{\beta j}}\right)\right\}\right)\mathbf{Z}\right)$$

$$W(\boldsymbol{\xi}) = W\left(\exp\left(\varepsilon \sum_{j=1}^{N}\left\{\frac{\partial^2 W}{\partial u_j \partial U_{\alpha j}} + \frac{\partial^2 W}{\partial u_j \partial U_{\beta j}} - \frac{\partial^2 W}{\partial u_j \partial U_{\alpha j}} - \frac{\partial^2 W}{\partial u_j \partial U_{\beta j}}\right\}\right)\right)$$

$$= W(\mathbf{Z})$$

Theorem 7. If the function $\mathcal{F}(\boldsymbol{\xi},\varepsilon)$ depends analytically on ε (i.e. admits Taylor series in the neighborhood of $\varepsilon = 0$) so that

$$\mathcal{F}(\boldsymbol{\xi},\varepsilon) = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \mathcal{F}_n(\boldsymbol{\xi}) \quad (15)$$

then the transform under the canonical mapping $\boldsymbol{\xi} = \exp(\varepsilon \mathfrak{L}_W)\mathbf{Z}$ is

$$\mathcal{F}(\boldsymbol{\xi},\varepsilon) = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \sum_{m=0}^{n} C_m^n \mathfrak{L}_W^m \mathcal{F}_{n-m}(\mathbf{Z}) \quad (16)$$

where $C_m^n = \dfrac{n!}{m!(n-m)!}$ is the binomial coefficient.

Proof: From theorem 4

$$\mathcal{F}_n(\boldsymbol{\xi}) = \exp(\varepsilon \mathfrak{L}_W)\mathcal{F}_n(\mathbf{Z})$$

$$= \sum_{m=0}^{n} \frac{\varepsilon^m}{m!} \mathfrak{L}_W^m \mathcal{F}_n(\mathbf{Z}) \quad (17)$$

so that

$$\mathcal{F}(\boldsymbol{\xi},\varepsilon) = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \sum_{m=1}^{n} \frac{\varepsilon^m}{m!} \mathfrak{L}_W^m \mathcal{F}_n(\mathbf{Z}) \quad (18)$$

set $m+n = N \Rightarrow N \ge 0, \quad 0 \le m \le N$, then

$$\mathcal{F}(\boldsymbol{\xi},\varepsilon) = \sum_{N\ge 0}^{\infty} \frac{\varepsilon^N}{m!(N-m)!} \sum_{m=1}^{N} \mathfrak{L}_W^m \mathcal{F}_{N-m}(\mathbf{Z}) \quad (19)$$

Theorem 8. The inverse of the fractional canonical mapping $\boldsymbol{\xi} = \exp(\varepsilon \mathfrak{L}_W)\mathbf{Z}$ is $\mathbf{Z} = \exp(\varepsilon \mathfrak{L}_{-W})\boldsymbol{\xi}$

Proof:

Let

$$\begin{aligned}\mathbf{Z} &= \exp\left(\varepsilon \mathfrak{L}_{W^*}\right)\boldsymbol{\xi} \\ &= \exp\left(\varepsilon \mathfrak{L}_{W^*}\right)\exp(\varepsilon \mathfrak{L}_W)\mathbf{Z} \\ &= \exp\left(\varepsilon\left(\mathfrak{L}_{W^*} + \mathfrak{L}_W\right)\right)\mathbf{Z},\end{aligned}$$

hence the the operator $\exp\left(\varepsilon\left(\mathfrak{L}_{W^*} + \mathfrak{L}_W\right)\right)$ must reduce to the identity transformation, i.e. $\mathfrak{L}_{W^*} + \mathfrak{L}_W = 0$, and therefore

$$W^* = -W$$

3. Composite Mappings

Define two successive transformations

$$\left(\mathbf{u},\mathbf{U}_{\alpha},\mathbf{U}_{\beta}\right) \overset{W}{\to} \left(\mathbf{u}^*,\mathbf{U}_{\alpha}^*,\mathbf{U}_{\beta}^*\right) \overset{W^*}{\to} \left(\mathbf{u}^{**},\mathbf{U}_{\alpha}^{**},\mathbf{U}_{\beta}^{**}\right)$$

with $W^*\left(\mathbf{u}^{**},\mathbf{U}_{\alpha}^{**},\mathbf{U}_{\beta}^{**}\right)$.

The composite transformation $\left(\mathbf{u},\mathbf{U}_{\alpha},\mathbf{U}_{\beta}\right) \to \left(\mathbf{u}^{**},\mathbf{U}_{\alpha}^{**},\mathbf{U}_{\beta}^{**}\right)$ is affected as follows:

$$\mathcal{F}\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta\right)$$

$$=\sum_{n\ge 0}^{\infty}\frac{\varepsilon^n}{n!}\mathfrak{L}_{W^*}^n\sum_{n\ge 0}^{\infty}\frac{\varepsilon^n}{n!}\mathfrak{L}_W^n\mathcal{F}\left(\mathbf{u}^{**},\mathbf{U}_\alpha^{**},\mathbf{U}_\beta^{**}\right)$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{\varepsilon^n}{n!}C_m^n\mathfrak{L}_{W^*}^m\mathfrak{L}_W^{n-m}\mathcal{F}\left(\mathbf{u}^{**},\mathbf{U}_\alpha^{**},\mathbf{U}_\beta^{**}\right)$$

$$\mathcal{F}\left(\mathbf{u},\mathbf{U}_\alpha^{**},\mathbf{U}_\beta^{**}\right)=\Big[1+\varepsilon\left(\mathfrak{L}_{W^*}+\mathfrak{L}_W\right)$$
$$+\frac{\varepsilon}{2!}\left(\mathfrak{L}_{W^*}^2+2\mathfrak{L}_{W^*}\mathfrak{L}_W+\mathfrak{L}_W^2\right)$$
$$+\ldots\Big]\mathcal{F}\left(\mathbf{u}^{**},\mathbf{U}_\alpha^{**},\mathbf{U}_\beta^{**}\right)$$
$$=\mathcal{F}\left(\mathbf{u}^{**},\mathbf{U}_\alpha^{**},\mathbf{U}_\beta^{**}\right)+\varepsilon\left(\mathcal{F};W+W^*\right)$$
$$+\frac{\varepsilon^2}{2!}\left(\mathfrak{L}_{W^*}^2+2\mathfrak{L}_{W^*}\mathfrak{L}_W+\mathfrak{L}_W^2\right)\mathcal{F}+\mathcal{O}\left(\varepsilon^3\right) \qquad (20)$$

Using the relation

$$\mathfrak{L}_{W^*}\mathfrak{L}_W=\mathfrak{L}_W\mathfrak{L}_{W^*}+\mathfrak{L}_{\left(W;W^*\right)},$$

the term factored by $\frac{\varepsilon^2}{2!}$ in the expansion of (20) becomes

$$\left(\mathfrak{L}_{W^*}^2+2\mathfrak{L}_{W^*}\mathfrak{L}_W+\mathfrak{L}_W^2\right)=\mathfrak{L}_{W^*}^2+\mathfrak{L}_W\mathfrak{L}_{W^*}$$
$$+\mathfrak{L}_{\left(W;W^*\right)}+\mathfrak{L}_{W^*}\mathfrak{L}_W+\mathfrak{L}_W^2$$
$$=\mathfrak{L}_W\left(\mathfrak{L}_W+\mathfrak{L}_{W^*}\right)+\mathfrak{L}_{\left(W;W^*\right)}+\mathfrak{L}_{W^*}\left(\mathfrak{L}_W+\mathfrak{L}_{W^*}\right)$$
$$=\left(\mathfrak{L}_W+\mathfrak{L}_{W^*}\right)\left(\mathfrak{L}_W+\mathfrak{L}_{W^*}\right)+\mathfrak{L}_{\left(W;W^*\right)}$$

therefore equation (20) retaining the terms up to the second order in ε becomes

$$\mathcal{F}\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta\right)=\mathcal{F}\left(\mathbf{u}^{**},\mathcal{F}\left(\mathbf{u}^{**},\mathbf{U}_\alpha^{**},\mathbf{U}_\beta^{**}\right)\right)$$
$$+\varepsilon\left(\mathcal{F};W+W^*\right)$$
$$+\frac{\varepsilon^2}{2!}\left(\left(\left(\mathcal{F};W+W^*\right);W+W^*\right)\right.$$
$$\left.+\left(\mathcal{F};W+W^*\right)\right) \qquad (21)$$

3.1. The Fractional Lie Operator

Any perturbation theory usually depends on a small parameter, ε, and the solution of the problem is known at $\varepsilon=0$ (or at any other specified value). In the Lie transformation theory just described, this parameter is not allowed for, however, this can be accomplished by introducing the fractional Lie operator

$$\Delta_W\mathcal{F}=\sum_{j=1}^{N}\left\{\frac{\partial\mathcal{F}}{\partial u_j}\left(\frac{\partial W}{\partial U_{\alpha j}}+\frac{\partial W}{\partial U_{\beta j}}\right)-\frac{\partial W}{\partial u_j}\left(\frac{\partial\mathcal{F}}{\partial U_{\alpha j}}+\frac{\partial\mathcal{F}}{\partial U_{\beta j}}\right)\right\}+\frac{\partial\mathcal{F}}{\partial\varepsilon} \qquad (22)$$

where

$$W=W\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta,\varepsilon\right),\ \mathcal{F}=\mathcal{F}\left(\mathbf{u},\mathbf{U}_\mathbf{a},\mathbf{U}_\mathbf{b},\varepsilon\right)$$

Similar to $\mathfrak{L}_W$, the Lie operator Δ_W has the following properties

$$\left.\begin{aligned}&\Delta_W\left(\alpha\mathcal{F}+\beta\mathcal{K}\right)=\alpha\Delta_W\left(\mathcal{F}\right)+\beta\Delta_W\left(\mathcal{K}\right)\\&\Delta_W\left(\mathcal{F}\cdot\mathcal{K}\right)=\mathcal{K}\cdot\Delta_W\mathcal{F}+\mathcal{F}\cdot\Delta_W\mathcal{K}\\&\Delta_W\left(\mathcal{F};\mathcal{K}\right)=\left(\mathcal{F};\Delta_W\mathcal{K}\right)+\left(\Delta_W\mathcal{F},\mathcal{K}\right)\\&\Delta_V\ \Delta_W=\Delta_W\ \Delta_V+\mathfrak{L}_{(W;V)}+\mathfrak{L}_{\left(\frac{\partial W}{\partial\varepsilon}-\frac{\partial V}{\partial\varepsilon}\right)}\end{aligned}\right\} \qquad (23)$$

The proofs of the first three properties are clear, to prove the last property we introduce the following Lemmas:

Lemma 9.

$$\frac{\partial}{\partial\varepsilon}\left(\mathfrak{L}_V\mathcal{F}\right)=\mathfrak{L}_V\frac{\partial\mathcal{F}}{\partial u_j}+\mathfrak{L}_{\frac{\partial V}{\partial\varepsilon}}\mathcal{F}$$

Proof:

$$\frac{\partial}{\partial\varepsilon}\left(\mathfrak{L}_V\mathcal{F}\right)=\frac{\partial}{\partial\varepsilon}\left(\sum_{j=1}^{N}\left\{\frac{\partial\mathcal{F}}{\partial u_j}\left(\frac{\partial V}{\partial U_{\alpha j}}+\frac{\partial V}{\partial U_{\beta j}}\right)\right.\right.$$
$$\left.\left.-\frac{\partial V}{\partial u_j}\left(\frac{\partial\mathcal{F}}{\partial U_{\alpha j}}+\frac{\partial\mathcal{F}}{\partial U_{\beta j}}\right)\right\}\right)$$
$$=\sum_{j=1}^{N}\left\{\frac{\partial}{\partial u_j}\left(\frac{\partial\mathcal{F}}{\partial\varepsilon}\right)\left(\frac{\partial V}{\partial U_{\alpha j}}+\frac{\partial V}{\partial U_{\beta j}}\right)\right.$$
$$\left.-\frac{\partial W}{\partial u_j}\left(\frac{\partial}{\partial U_{\alpha j}}\left(\frac{\partial\mathcal{F}}{\partial\varepsilon}\right)+\frac{\partial}{\partial U_{\beta j}}\left(\frac{\partial\mathcal{F}}{\partial\varepsilon}\right)\right)\right\}$$

$$+\sum_{j=1}^{N}\left\{\frac{\partial\mathcal{F}}{\partial u_j}\left(\frac{\partial}{\partial U_{\alpha j}}\left(\frac{\partial V}{\partial\varepsilon}\right)+\frac{\partial}{\partial U_{\beta j}}\left(\frac{\partial V}{\partial\varepsilon}\right)\right)\right.$$

$$\left.-\frac{\partial}{\partial u_j}\left(\frac{\partial V}{\partial\varepsilon}\right)\left(\frac{\partial\mathcal{F}}{\partial U_{\alpha j}}+\frac{\partial\mathcal{F}}{\partial U_{\beta j}}\right)\right\}$$

$$\frac{\partial}{\partial\varepsilon}\left(\mathfrak{L}_V\mathcal{F}\right)=\mathfrak{L}_V\frac{\partial\mathcal{F}}{\partial u_j}+\mathfrak{L}_{\frac{\partial V}{\partial\varepsilon}}\mathcal{F}$$

Lemma 10.

$$\mathfrak{L}_{-V}\mathcal{F}=-\mathfrak{L}_V\mathcal{F}$$

Proof:

$$\mathfrak{L}_{-V}\mathcal{F}=\sum_{j=1}^{N}\left\{\frac{\partial\mathcal{F}}{\partial u_j}\left(\frac{\partial(-V)}{\partial_a D_t^{\alpha}u_j}+\frac{\partial(-V)}{\partial_t D_b^{\beta}u_j}\right)\right.$$

$$\left.-\frac{\partial(-V)}{\partial u_j}\left(\frac{\partial\mathcal{F}}{\partial_a D_t^{\alpha}u_j}+\frac{\partial\mathcal{F}}{\partial_t D_b^{\beta}u_j}\right)\right\}$$

$$=\sum_{j=1}^{N}\left\{-\frac{\partial\mathcal{F}}{\partial u_j}\left(\frac{\partial(V)}{\partial_a D_t^{\alpha}u_j}+\frac{\partial(V)}{\partial_t D_b^{\beta}u_j}\right)\right.$$

$$\left.+\frac{\partial(V)}{\partial u_j}\left(\frac{\partial\mathcal{F}}{\partial_a D_t^{\alpha}u_j}+\frac{\partial\mathcal{F}}{\partial_t D_b^{\beta}u_j}\right)\right\}$$

$$=-\sum_{j=1}^{N}\left\{\frac{\partial\mathcal{F}}{\partial u_j}\left(\frac{\partial(V)}{\partial U_{\alpha j}}+\frac{\partial(V)}{\partial U_{\beta j}}\right)\right.$$

$$\left.-\frac{\partial(V)}{\partial u_j}\left(\frac{\partial\mathcal{F}}{\partial U_{\alpha j}}+\frac{\partial\mathcal{F}}{\partial U_{\beta j}}\right)\right\}=-\mathfrak{L}_V\mathcal{F}$$

Lemma 11.

$$\mathfrak{L}_{\frac{\partial W}{\partial\varepsilon}}\mathcal{F}-\mathfrak{L}_{\frac{\partial V}{\partial\varepsilon}}\mathcal{F}=\mathfrak{L}_{\left(\frac{\partial W}{\partial\varepsilon}-\frac{\partial V}{\partial\varepsilon}\right)}\mathcal{F}$$

Proof: Using the previous Lemma

$$\mathfrak{L}_{\frac{\partial W}{\partial\varepsilon}}\mathcal{F}-\mathfrak{L}_{\frac{\partial V}{\partial\varepsilon}}\mathcal{F}=\left(\mathfrak{L}_{\frac{\partial W}{\partial\varepsilon}}+\mathfrak{L}_{-\frac{\partial V}{\partial\varepsilon}}\right)\mathcal{F}$$

$$\left(\mathfrak{L}_{\frac{\partial W}{\partial\varepsilon}}+\mathfrak{L}_{-\frac{\partial V}{\partial\varepsilon}}\right)\mathcal{F}=\sum_{j=1}^{N}\left\{\frac{\partial\mathcal{F}}{\partial u_j}\left(\frac{\partial\left(\frac{\partial W}{\partial\varepsilon}\right)}{\partial U_{\alpha j}}\right.\right.$$

$$\left.\left.+\frac{\partial\left(\frac{\partial W}{\partial\varepsilon}\right)}{\partial U_{\beta j}}-\frac{\partial\left(\frac{\partial V}{\partial\varepsilon}\right)}{\partial U_{\alpha j}}-\frac{\partial\left(\frac{\partial V}{\partial\varepsilon}\right)}{\partial U_{\beta j}}\right)\right.$$

$$\left.-\left\{\frac{\partial}{\partial u_j}\left(\frac{\partial W}{\partial\varepsilon}-\frac{\partial V}{\partial\varepsilon}\right)\left(\frac{\partial\mathcal{F}}{\partial U_{\alpha j}}+\frac{\partial\mathcal{F}}{\partial U_{\beta j}}\right)\right\}\right.$$

$$\left(\mathfrak{L}_{\frac{\partial W}{\partial\varepsilon}}+\mathfrak{L}_{-\frac{\partial V}{\partial\varepsilon}}\right)\mathcal{F}=$$

$$=\sum_{j=1}^{N}\left\{\frac{\partial\mathcal{F}}{\partial u_j}\left[\frac{\partial}{\partial U_{\alpha j}}\left(\frac{\partial W}{\partial\varepsilon}-\frac{\partial v}{\partial\varepsilon}\right)\right.\right.$$

$$\left.+\frac{\partial}{\partial U_{\beta j}}\left(\frac{\partial W}{\partial\varepsilon}-\frac{\partial V}{\partial\varepsilon}\right)\right]$$

$$\left.-\left\{\left(\frac{\partial W}{\partial u_j}-\frac{\partial V}{\partial u_j}\right)\left(\frac{\partial\mathcal{F}}{\partial U_{\alpha j}}+\frac{\partial\mathcal{F}}{\partial U_{\beta j}}\right)\right\}\right\}$$

$$=\mathfrak{L}_{\left(\frac{\partial W}{\partial\varepsilon}-\frac{\partial V}{\partial\varepsilon}\right)}\mathcal{F}$$

Now we are going to prove the last property mentioned above

$$\Delta_V\,\Delta_W=\Delta_W\,\Delta_V+\mathfrak{L}_{(W;V)}+\mathfrak{L}_{\left(\frac{\partial W}{\partial\varepsilon}-\frac{\partial V}{\partial\varepsilon}\right)}$$

Proof:

$$\Delta_V\,\Delta_W=\left(\mathfrak{L}_V+\frac{\partial}{\partial\varepsilon}\right)\left(\mathfrak{L}_W+\frac{\partial}{\partial\varepsilon}\right)$$

$$=\mathfrak{L}_W\mathfrak{L}_V+\mathfrak{L}_{(W;V)}+\mathfrak{L}_V\frac{\partial}{\partial\varepsilon}$$

$$+\mathfrak{L}_W\frac{\partial}{\partial\varepsilon}+\mathfrak{L}_{\frac{\partial W}{\partial\varepsilon}}+\frac{\partial^2}{\partial\varepsilon^2}$$

$$=\left(\mathfrak{L}_W\mathfrak{L}_V+\mathfrak{L}_W\frac{\partial}{\partial\varepsilon}+\frac{\partial}{\partial\varepsilon}\mathfrak{L}_V+\frac{\partial^2}{\partial\varepsilon^2}\right)$$

$$+\mathfrak{L}_{(W;V)}+\mathfrak{L}_{\frac{\partial W}{\partial\varepsilon}}-\mathfrak{L}_{\frac{\partial V}{\partial\varepsilon}}$$

$$= \Delta_W \ \Delta_V \ + \mathfrak{L}_{(W\,;V\,)} + \mathfrak{L}_{\left(\frac{\partial W}{\partial \varepsilon} - \frac{\partial V}{\partial \varepsilon}\right)}$$

The n th iterate of the Lie operator Δ_W^n is defined as Lie derivative $\mathfrak{L}_W^n$ with the corresponding properties

$$\left.\begin{aligned} &\Delta_W^n\left(\alpha\mathcal{F} + \beta\mathcal{K}\right) = \alpha\Delta_W^n\left(\mathcal{F}\right) + \beta\Delta_W^n\left(\mathcal{K}\right) \\ &\Delta_W^n\left(\mathcal{F}\cdot\mathcal{K}\right) = \sum_{m=0}^{n} C_m^n \Delta_W^m \mathcal{F}\cdot\Delta_W^{n-m}\mathcal{K} \\ &\Delta_W^n\left(\mathcal{F};\mathcal{K}\right) = \sum_{m=0}^{n} C_m^n \left(\Delta_W^m \mathcal{F};\Delta_W^{n-m}\mathcal{K}\right) \end{aligned}\right\} \qquad (24)$$

These relations can be obtained from the successive applications of the operator Δ_W

4. Lie Transform Depending on a parameter

Consider the mapping

$$E_W : \mathcal{F} \to E_W\mathcal{F} = \sum_{n=0} \frac{\varepsilon^n}{n!}\mathcal{F}^{(n)}\left(\mathbf{u}, \mathbf{U}_\alpha, \mathbf{U}_\beta, 0\right) \qquad (25)$$

where

$$\mathcal{F}^{(n)}\left(\mathbf{u}, \mathbf{U}_\alpha, \mathbf{U}_\beta, 0\right) = \Delta_W^n \mathcal{F}\left(\mathbf{u}, \mathbf{U}_\alpha, \mathbf{U}_\beta, 0\right)\Big|_{\varepsilon=0}$$

This mapping is called the Lie transform generated by W. Obviously if there exists a finite quantity k such that

$$\mathcal{F}^{(n)}\left(\mathbf{u}, \mathbf{U}_\alpha, \mathbf{U}_\beta, 0\right) < k^n$$

in some neighborhood of a point $\left(\mathbf{u}_0, \mathbf{U}_{\alpha 0}, \mathbf{U}_{\beta 0}\right)$, the series (25) converges. The following relations are satisfied by E_W :

$$E_W\left(\alpha\mathcal{F} + \beta\mathcal{K}\right) = \alpha E_W\mathcal{F} + \beta E_W\mathcal{K} \qquad (26)$$

$$E_W\left(\mathcal{F}\cdot\mathcal{K}\right) = E_W\mathcal{F}\cdot E_W\mathcal{K} \qquad (27)$$

$$E_W\left(\mathcal{F};\mathcal{K}\right) = \left(E_W\mathcal{F}; E_W\mathcal{K}\right) \qquad (28)$$

Proof: The first relation (26) follows directly. Now we are going to prove the relation (27)

$$E_W\left(\mathcal{F}\cdot\mathcal{K}\right) = \sum_{n=0}^{\infty}\frac{\varepsilon^n}{n!}\sum_{m=0}^{n} C_m^n \Delta_W^m \mathcal{F}\Delta_W^{n-m}\mathcal{K}\Big|_{\varepsilon=0}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{\varepsilon^m}{m!}\Delta_W^m\mathcal{F}\Big|_{\varepsilon=0}\frac{\varepsilon^{n-m}}{(n-m)!}\Delta_W^{n-m}\mathcal{K}\Big|_{\varepsilon=0}$$

$$E_W\left(\mathcal{F}\cdot\mathcal{K}\right) = \sum_{n=0}^{n}\sum_{m=0}^{n} a_m b_{n-m}$$

where

$$a_r = \frac{\varepsilon^r}{r!}\Delta_W^r\mathcal{F}\Big|_{\varepsilon=0}, \qquad b_r = \frac{\varepsilon r}{r!}\Delta_W^r\mathcal{K}\Big|_{\varepsilon=0}$$

then

$$E_W\left(\mathcal{F}\cdot\mathcal{K}\right) = \sum_{n=0}^{\infty} a_m \sum_{m=0}^{\infty} b_n = E_W\mathcal{F}\cdot E_W\mathcal{K}$$

Now we are going to prove the relation (28)

$$E_W\left(\mathcal{F};\mathcal{K}\right) = \sum_{n=0}^{\infty}\frac{\varepsilon^n}{n!}\sum_{m=0}^{\infty} C_m^n\Delta_W^n\left(\mathcal{F};\mathcal{K}\right)\Big|_{\varepsilon=0}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{\varepsilon^m}{m!}\Delta_W^m\mathcal{F}^{(m)}; \frac{\varepsilon^{n-m}}{(n-m)!}\mathcal{K}^{(n-m)}\right)\Bigg|_{\varepsilon=0}$$

$$E_W\left(\mathcal{F};\mathcal{K}\right) = \sum_{j=0}^{\infty}\Bigg[\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{\varepsilon^m}{m!}\frac{\partial\mathcal{F}^{(m)}}{\partial u_j}\right)$$

$$\times\left(\frac{\varepsilon^{n-m}}{(n-m)!}\left(\frac{\partial\mathcal{K}^{(n-m)}}{\partial U_{\alpha j}} + \frac{\partial\mathcal{K}^{(n-m)}}{\partial U_{\beta j}}\right)\right)$$

$$-\left(\frac{\varepsilon^m}{m!}\left(\frac{\partial\mathcal{F}^{(m)}}{\partial U_{\alpha j}} + \frac{\partial\mathcal{F}^{(m)}}{\partial U_{\beta j}}\right)\right)\left(\frac{\varepsilon^{n-m}}{(n-m)!}\frac{\partial\mathcal{K}^{(n-m)}}{\partial u_j}\right)\Bigg]_{\varepsilon=0}$$

$$= \sum_{j=0}^{\infty}\Bigg[\sum_{n=0}^{\infty}\left(\frac{\varepsilon^n}{n!}\frac{\partial\mathcal{F}^{(n)}}{\partial u_j}\right)_{n=0}^{\infty}\left(\frac{\varepsilon^n}{n!}\left(\frac{\partial\mathcal{K}^{(n)}}{\partial U_{\alpha j}} + \frac{\partial\mathcal{K}^{(n)}}{\partial U_{\beta j}}\right)\right)$$

$$-\sum_{n=0}^{\infty}\left(\frac{\varepsilon^n}{n!}\left(\frac{\partial\mathcal{F}^{(n)}}{\partial U_{\alpha j}} + \frac{\partial\mathcal{F}^{(n)}}{\partial U_{\beta j}}\right)\right)_{n=0}^{\infty}\left(\frac{\varepsilon^n}{n!}\frac{\partial\mathcal{K}^{(n)}}{\partial u_j}\right)\Bigg]_{\varepsilon=0}$$

$$E_W\left(\mathcal{F};\mathcal{K}\right) = \sum_{j=0}^{\infty}\Bigg[\frac{\partial\left(E_W\mathcal{F}\right)}{\partial u_j}\left(\frac{\partial\left(E_W\mathcal{K}\right)}{\partial U_{\alpha j}}\right.$$

$$\left. + \frac{\partial\left(E_W\mathcal{K}\right)}{\partial U_{\beta j}}\right) - \left(\frac{\partial\left(E_W\mathcal{F}\right)}{\partial U_{\alpha j}} + \frac{\partial\left(E_W\mathcal{F}\right)}{\partial U_{\beta j}}\right)$$

$$\times\frac{\partial\left(E_W\mathcal{K}\right)}{\partial u_j}\Bigg] = \left(E_W\mathcal{F}; E_W\mathcal{K}\right)$$

Theorem 12. The transformation $(\mathbf{Z},\varepsilon)\to\Phi$, defined by $\mathbf{\Phi}=E_W\mathbf{Z}$ is completely canonical

Proof:

$$\left(\Phi_\mu;\Phi_\nu\right)=\left(E_W Z_\mu;E_W Z_\nu\right)=E_W\left(Z_\mu;Z_\nu\right)$$

$$=E_W\sum_{j=1}^{N}\left\{\frac{\partial u_\mu}{\partial u_j}\left(\frac{\partial U_{\alpha\nu}}{\partial U_{\alpha j}}+\frac{\partial U_{\beta\nu}}{\partial U_{\beta j}}\right)\right.$$

$$\left.-\frac{\partial}{\partial u_j}\left(U_{\alpha\mu}+U_{\beta\mu}\right)\left(\frac{\partial u_\nu}{\partial U_{\alpha j}}+\frac{\partial u_\nu}{\partial U_{\beta j}}\right)\right\}$$

$$\left(\Phi_\mu;\Phi_\nu\right)=E_W J^{\mu\nu} \tag{29}$$

Theorem 13. The transformation $\mathbf{\Phi}=E_W\mathbf{Z}$ is the solution of the Hamiltonian system

$$\left.\begin{aligned}\frac{d\boldsymbol{\xi}}{d\varepsilon}&=\frac{\partial W(\Phi,\boldsymbol{\varepsilon})}{\partial\Xi_\alpha}+\frac{\partial W(\Phi,\boldsymbol{\varepsilon})}{\partial\Xi_\beta},\\ \frac{d\left(\Xi_\alpha+\Xi_\beta\right)}{d\varepsilon}&=-\frac{\partial W(\Phi,\boldsymbol{\varepsilon})}{\partial\boldsymbol{\xi}}\end{aligned}\right\} \tag{30}$$

with the initial conditions

$$\mathbf{\Phi}\left(\boldsymbol{\xi},\Xi_{\mathbf{a}},\Xi_{\mathbf{b}}\right)=\mathbf{Z}\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta\right)\ \text{at}\ \varepsilon=0$$

Proof: To prove this theorem we shall prove that the expansion of the Φ of the Hamiltonian system (30) in powers of ε results in the expressions $\mathbf{\Phi}=E_W\mathbf{Z}$. Assuming $\mathbf{\Phi}(\mathbf{Z},\varepsilon)$ real analytic function

$$\mathbf{\Phi}=\sum_{n=0}^{\infty}\frac{\varepsilon^n}{n!}\frac{d^n\Phi\left(\boldsymbol{\xi},\Xi_{\mathbf{a}},\Xi_{\mathbf{b}}\right)}{d\varepsilon^n}\Bigg|_{\varepsilon=0}$$

but

$$\frac{d\mathbf{\Phi}}{d\varepsilon}=\frac{\partial\mathbf{\Phi}}{\partial\varepsilon}+\left(\frac{\partial\mathbf{\Phi}}{\partial\Xi_\alpha}+\frac{\partial\mathbf{\Phi}}{\partial\Xi_\beta}\right)\frac{\partial\left(\Xi_\alpha+\Xi_\beta\right)}{\partial\varepsilon}+\frac{\partial\mathbf{\Phi}}{\partial\boldsymbol{\xi}}\frac{\partial\boldsymbol{\xi}}{\partial\varepsilon}$$

$$\frac{d\mathbf{\Phi}}{d\varepsilon}=\frac{\partial\mathbf{\Phi}}{\partial\varepsilon}-\left(\frac{\partial\mathbf{\Phi}}{\partial\Xi_\alpha}+\frac{\partial\mathbf{\Phi}}{\partial\Xi_\beta}\right)\frac{\partial W(\Phi,\boldsymbol{\varepsilon})}{\partial\boldsymbol{\xi}}+\frac{\partial\mathbf{\Phi}}{\partial\boldsymbol{\xi}}\left(\frac{\partial W(\Phi,\boldsymbol{\varepsilon})}{\partial\Xi_\alpha}+\frac{\partial W(\Phi,\boldsymbol{\varepsilon})}{\partial\Xi_\beta}\right)$$

$$\frac{d\mathbf{\Phi}}{d\varepsilon}=\frac{\partial\mathbf{\Phi}}{\partial\varepsilon}+\frac{\partial\mathbf{\Phi}}{\partial\boldsymbol{\xi}}\left(\frac{\partial W(\Phi,\boldsymbol{\varepsilon})}{\partial\Xi_\alpha}+\frac{\partial W(\Phi,\boldsymbol{\varepsilon})}{\partial\Xi_\beta}\right)-\frac{\partial W(\Phi,\boldsymbol{\varepsilon})}{\partial\boldsymbol{\xi}}\left(\frac{\partial\mathbf{\Phi}}{\partial\Xi_\alpha}+\frac{\partial\mathbf{\Phi}}{\partial\Xi_\beta}\right)$$

$$\frac{d\mathbf{\Phi}}{d\varepsilon}=\frac{\partial\mathbf{\Phi}}{\partial\varepsilon}+\mathfrak{L}_W\mathbf{\Phi}=\Delta_W\mathbf{\Phi} \tag{31}$$

Therefore

$$\mathbf{\Phi}=\sum_{n=0}^{\infty}\frac{\varepsilon^n}{n!}\Delta_W^n\,\mathbf{\Phi}\Big|_{\varepsilon=0}=\sum_{n=0}^{\infty}\frac{\varepsilon^n}{n!}\Delta_W^n\,\mathbf{Z}\Big|_{\varepsilon=0}=E_W(\mathbf{Z}) \tag{32}$$

Theorem 14. The image of real analytic function $\mathcal{F}\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta,\varepsilon\right)$ under the mapping $\mathbf{\Phi}=E_W\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta\right)$ is

$$\mathcal{F}\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta,\varepsilon\right)=E_W\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta,\varepsilon\right) \tag{33}$$

Proof: The proof follows directly from (31) since

$$\frac{d\mathcal{F}}{d\varepsilon}=\frac{d\mathcal{F}}{d\mathbf{\Phi}}\frac{\partial\mathbf{\Phi}}{\partial\varepsilon}=\frac{d\mathcal{F}}{d\mathbf{\Phi}}\Delta_W\mathbf{\Phi}=\Delta_W\mathcal{F}\left(\mathbf{\Phi},\boldsymbol{\varepsilon}\right)$$

$$\frac{d\mathcal{F}}{d\varepsilon}\Big|_{\varepsilon=0}=\Delta_W\mathcal{F}\left(\mathbf{F},\boldsymbol{\varepsilon}\right)\Big|_{\varepsilon=0}=\Delta_W\mathcal{F}\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta,\varepsilon\right)\Big|_{\varepsilon=0}$$

$$\mathcal{F}\left(\mathbf{\Phi},\boldsymbol{\varepsilon}\right)=\sum_{n=0}^{\infty}\frac{\varepsilon^n}{n!}\Delta_W^n\,\mathcal{F}\Big|_{\varepsilon=0}=E_W\,\mathcal{F}\left(\mathbf{u},\mathbf{U}_{\mathbf{a}},\mathbf{U}_{\mathbf{b}},\varepsilon\right)$$

5. Recurrence Relations

When both $W\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta,\varepsilon\right)$ and $\mathcal{F}\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta,\varepsilon\right)$ can be expanded in power series in ε, the coefficients $\mathcal{F}^{(n)}\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta,0\right)$ can be constructed recurrsively. Using

$$W\left(\mathbf{u},\mathbf{U_a},\mathbf{U_b},\boldsymbol{\varepsilon}\right)=\sum_{n=0}^{\infty}\frac{\varepsilon^n}{n!}W_{n+1}\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta\right) \quad (34)$$

$$\mathcal{F}\left(\mathbf{u},\mathbf{U_a},\mathbf{U_b},\boldsymbol{\varepsilon}\right)=\sum_{n=0}^{\infty}\frac{\varepsilon^n}{n!}\mathcal{F}_n\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta\right) \quad (35)$$

to simplify the notation set $\mathfrak{L}_{Wp}=\mathfrak{L}p,\quad p\geq 1,$ then

$$\frac{\partial}{\partial\varepsilon}\mathcal{F}\left(\mathbf{u},\mathbf{U_a},\mathbf{U_b},\boldsymbol{\varepsilon}\right)=\sum_{n=0}^{\infty}\frac{\varepsilon^n}{n!}\mathcal{F}_{n+1}\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta\right) \quad (36)$$

$$\mathfrak{L}_W\mathcal{F}\left(\mathbf{u},\mathbf{U_a},\mathbf{U_b},\boldsymbol{\varepsilon}\right)=\sum_{n=0}^{\infty}\frac{\varepsilon^n}{n!}\mathfrak{L}_{n+1}\mathcal{F}\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta\right)$$
$$=\sum_{n=0}^{\infty}\frac{\varepsilon^n}{n!}\mathfrak{L}_{n+1}\sum_{n=0}^{\infty}\frac{\varepsilon^n}{n!}\mathcal{F}_n\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta\right)$$
$$=\sum_{n=0}^{\infty}\frac{\varepsilon^n}{n!}\sum_{m=0}^{n}C_m^n\mathfrak{L}_{m+1}\mathcal{F}_{n-m}\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta\right) \quad (37)$$

We can thus represent $\Delta_W\mathcal{F}\left(\mathbf{u},\mathbf{U_a},\mathbf{U_b},\boldsymbol{\varepsilon}\right)$ by the series

$$\Delta_W\mathcal{F}\left(\mathbf{u},\mathbf{U_a},\mathbf{U_b},\boldsymbol{\varepsilon}\right)=\sum_{n=0}^{\infty}\frac{\varepsilon^n}{n!}\mathcal{F}_n^{(1)}\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta\right) \quad (38)$$

with

$$\mathcal{F}_n^{(1)}\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta\right)=\mathcal{F}_{n+1}\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta\right)+\sum_{m=0}^{n}C_m^n\mathfrak{L}_{m+1}\mathcal{F}_{n-m}\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta\right) \quad (39)$$

Consequently

$$\mathcal{F}^{(1)}\left(\mathbf{u},\mathbf{U_a},\mathbf{U_b},\mathbf{0}\right)=\mathcal{F}_o^{(1)}\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta\right)$$
$$=\mathcal{F}_1+\mathfrak{L}_1\mathcal{F}_0$$
$$=\mathcal{F}_1+\left(\mathcal{F}_0;W_1\right) \quad (40)$$

Similarly setting

$$\Delta_W^2\mathcal{F}\left(\mathbf{u},\mathbf{U_a},\mathbf{U_b},\boldsymbol{\varepsilon}\right)=\sum_{n=0}^{\infty}\frac{\varepsilon^n}{n!}\mathcal{F}_n^{(2)}\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta\right) \quad (41)$$

We find

$$\mathcal{F}_n^{(2)}\left(\mathbf{u},\mathbf{U_a},\mathbf{U_b},\mathbf{0}\right)=\mathcal{F}_{n+1}^{(1)}\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta\right)+\sum_{m=0}^{n}C_m^n\mathfrak{L}_{m+1}\mathcal{F}_{n-m}^{(1)}\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta\right) \quad (42)$$

Consequently

$$\mathcal{F}^{(2)}\left(\mathbf{u},\mathbf{U_a},\mathbf{U_b},\mathbf{0}\right)=\mathcal{F}_0^{(2)}\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta\right)$$
$$=\mathcal{F}_1^{(1)}+\mathfrak{L}_1\mathcal{F}_0^{(1)}$$
$$=\mathcal{F}_1+\left(\mathcal{F}_0;W_1\right) \quad (43)$$

Using (39) and (40) we obtain

$$\mathcal{F}_0^{(2)}\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta\right)=\mathcal{F}_2+2\left(\mathcal{F}_1;W_1\right)+\left(\mathcal{F}_0;W_2\right)+\left(\left(\mathcal{F}_0;W_1\right);W_1\right) \quad (44)$$

Generally

$$\Delta_W^k\mathcal{F}\left(\mathbf{u},\mathbf{U_a},\mathbf{U_b},\boldsymbol{\varepsilon}\right)=\sum_{n=0}^{\infty}\frac{\varepsilon^n}{n!}\mathcal{F}_n^{(k)}\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta\right)$$
$$\mathcal{F}_n^{(k)}\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta\right)=\mathcal{F}_{n+1}^{(k-1)}\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta\right)$$
$$+\sum_{m=0}^{n}C_m^n\mathfrak{L}_{m+1}\mathcal{F}_{n-m}^{(k-1)}\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta\right) \quad (45)$$

$$\mathcal{F}^k\left(\mathbf{u},\mathbf{U_a},\mathbf{U_b},0\right)=\mathcal{F}_0^{(k)}\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta\right)$$
$$=\mathcal{F}_1^{(k-1)}\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta\right)$$
$$+\mathfrak{L}_1\mathcal{F}^{(k-1)}\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta\right) \quad (46)$$

This can be represented by the following recursive diagram.

5.1. The Recursive Diagram

$$\begin{array}{lllll}
\mathcal{F}_0=\mathcal{F}^{(0)} & & & & \\
\downarrow & & & & \\
\mathcal{F}_1 \rightarrow & \mathcal{F}_0^{(1)}=\mathcal{F}^{(1)} & & & \\
\downarrow & \downarrow & & & \\
\mathcal{F}_2 \rightarrow & \mathcal{F}_1^{(1)} \rightarrow & \mathcal{F}_0^{(2)}=\mathcal{F}^{(2)} & & \\
\downarrow & \downarrow & \downarrow & & \\
\mathcal{F}_3 \rightarrow & \mathcal{F}_2^{(1)} \rightarrow & \mathcal{F}_1^{(2)} \rightarrow & \mathcal{F}_0^{(3)}=\mathcal{F}^{(3)} & \\
\downarrow & \downarrow & & & \\
\mathcal{F}_4 \rightarrow & & & &
\end{array}$$

any element is obtained from those in the column just to the left and in the same row and those above the considered element, e.g.

$$\mathcal{F}_2^{(1)}=\mathcal{F}_3+\mathfrak{L}_1\mathcal{F}_2+2\mathfrak{L}_2\mathcal{F}_1+\mathfrak{L}_3\mathcal{F}_0$$

$$\mathcal{F}_1^{(2)} = \mathcal{F}_2^{(1)} + \mathfrak{L}_1\mathcal{F}_1^{(1)} + \mathfrak{L}_2\mathcal{F}_0^{(1)}$$
$$\mathcal{F}^{(3)} = \mathcal{F}_1^{(2)} + \mathfrak{L}_1\mathcal{F}_0^{(2)}$$

Then the function $\mathcal{F}^{(n)}$ can be constructed recursively as Deprit did in [2] as follows

$$\Delta_w^k\mathcal{F} = \sum_{n\geq 0}^{\infty}\frac{\varepsilon^n}{n!}\mathcal{F}_n^{(k)}\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta\right) \quad (47)$$

$$\mathcal{F}_n^{(k)}\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta\right) = \mathcal{F}_{n+1}^{(k-1)}\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta\right) + \sum_{m=0}^{n}C_m^n L_{m+1}\mathcal{F}_{n-m}^{(k-1)}\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta\right) \quad (48)$$

$$\mathcal{F}^{(k)}\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta;0\right) = \mathcal{F}_0^{(k)}\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta\right) = \mathcal{F}_1^{(k-1)} + L_1\mathcal{F}^{(k+1)} \quad (49)$$

Now we can consider the transformation

$$\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta;\varepsilon\right)\rightarrow\left(\mathbf{u}^*,\mathbf{U}_\alpha^*,\mathbf{U}_\beta^*\right),$$

the fundamental transformation equation can be obtained as

$$\mathbf{u}^* = \mathbf{u} + \sum_{n\geq 1}^{\infty}\frac{\varepsilon^n}{n!}\mathbf{u}^{(n)} \quad (50)$$

$$\mathbf{U}_\alpha^* + \mathbf{U}_\beta^* = \mathbf{U}_\alpha + \mathbf{U}_\beta + \sum_{n\geq 1}^{\infty}\frac{\varepsilon^n}{n!}\left(\mathbf{U}_\alpha^{(n)} + \mathbf{U}_\beta^{(n)}\right) \quad (51)$$

$$\mathcal{F}\left(\mathbf{u}^*,\mathbf{U}_\alpha^*,\mathbf{U}_\beta^*;\varepsilon\right) = \mathcal{F}\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta\right) + \sum_{n\geq 1}^{\infty}\frac{\varepsilon^n}{n!}\mathcal{F}^{(n)}\left(\mathbf{u},\mathbf{U}_\alpha,\mathbf{U}_\beta\right) \quad (52)$$

The inverse transformation can be written as

$$\mathbf{u} = \mathbf{u}^* + \sum_{n\geq 1}^{\infty}\frac{\varepsilon^n}{n!}\mathbf{u}^{*(n)}\left(\mathbf{u}^*,\mathbf{U}_\alpha^*,\mathbf{U}_\beta^*\right) \quad (53)$$

$$\mathbf{U}_\alpha + \mathbf{U}_\beta = \mathbf{U}_\alpha^* + \mathbf{U}_\beta^* \quad (54)$$

5.2. Simplified General Algorithm

To simplify the algorithm for evaluating the transformation and its inverse, set equation (48) in the form

$$\mathcal{F}_n^{(k)} = \mathcal{F}_{n-1}^{(k+1)} - \sum_{m=0}^{n-1}C_m^n\mathfrak{L}_{m+1}\mathcal{F}_{n-m-1}^{(k)},\ n\geq 1; k\geq 0 \quad (55)$$

We then proceed to write the right-hand side of equation (55) such that $\mathcal{F}_n^{(k)}$ may be obtained in terms of $\mathcal{F}^{(k+n)}$, $\mathcal{F}^{(k+n-1)}$,... $\mathcal{F}^{(k)}$

Thus it can be assumed

$$\mathcal{F}_n^{(k)} = \mathcal{F}^{(k+n)} - \sum_{j=1}^{n}C_j^{n-1}G_j\mathcal{F}^{(k+n-j)},\ n\geq 1; k\geq 0 \quad (56)$$

where G_j is a linear operator which is a function of $\mathfrak{L}_j,\ \mathfrak{L}_{j-1},\ \mathfrak{L}_{j-2},........,\mathfrak{L}_1$

Substituting equation (56) into equation (55) yields the following recursive relationship;

$$G_j = \mathfrak{L}_j - \sum_{m=0}^{j-2}C_m^{j-1}\mathfrak{L}_{n-1}G_{j-m-1},\ n\geq 1; k\geq 0 \quad (57)$$

Successive application of equations (55) and (56) yield the required relations for the vector transformation.

5.3. Vector Transformations

The vector functions $\mathbf{u}^{(n)}$ and ${}_aD_t^\alpha\mathbf{u}^{(n)} + {}_tD_b^\beta\mathbf{u}^{(n)}$ required to implement the transformation (50), (51), (53) and (54) are given for $n\geq 1$ by

$$\left.\begin{aligned}\mathbf{u}^{(n)} &= \frac{\partial W_n}{\partial\mathbf{U}_\alpha} + \frac{\partial W_n}{\partial\mathbf{U}_\beta} + \sum_{j=1}^{n-1}C_j^{n-1}G_j^{(n-j)}\mathbf{u},\\ \mathbf{U}_\alpha^{(n)} + \mathbf{U}_\beta^{(n)} &= \frac{\partial W_n}{\partial\mathbf{u}} + \sum_{j=1}^{n-1}C_j^{n-1}G_j\left(\mathbf{U}_\alpha^{(n-j)} + \mathbf{U}_\beta^{(n-j)}\right)\end{aligned}\right\} \quad (58)$$

For the inverse transformation (53) and (54) we find that

$$\left.\begin{aligned}\mathbf{u}^{*(n)} &= -\mathbf{u}^{(n)} + \sum_{j=1}^{n-1}C_j^{n-1}G_j^{(n-j)}\mathbf{u},\\ \mathbf{U}_\alpha^{*(n)} + \mathbf{U}_\beta^{*(n)} &= -\left(\mathbf{U}_\alpha^{(n)} + \mathbf{U}_\beta^{(n)}\right)\\ &+ \sum_{j=1}^{n-1}C_j^{n-1}G_j\left(\mathbf{U}_\alpha^{(n-j)} + \mathbf{U}_\beta^{(n-j)}\right)\end{aligned}\right\} \quad (59)$$

(54)

6. Conclusion

We presented the right and the left Riemann-Liouville fractional differintegral operator. Also, we defined the fractional Poisson bracket over the fractional phase space as 3N state vector. As usual in celestial mechanics literature we defined the fractional Lie derivative $\mathfrak{L}_W\mathcal{F}$ generated by W as the fractional Poisson bracket $\left(\mathcal{F};W\right)$.

The properties of the fractional Lie derivative are outlined and proved. We defied the exponential operator over the fractional phase space. A theorem of a transformation using the exponential operator over the fractional phase space is proved to be completely canonical. The conserved generator W under this canonical mapping is proved in a corollary. A Theorem for the inverse fractional canonical mapping is proved. The composite mappings of two successive transformations is defined. The fractional Lie operator $\Delta_W \mathcal{F}$ and its properties are introduced. Some useful lemmas on this operator are proved as a preceding step to its n^th^ iterate. Lie transform depending on a parameter over the fractional phase space

$$E_W : \mathcal{F} \to E_W \mathcal{F} = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \mathcal{F}^{(n)} \left(\mathbf{u}, \mathbf{U}_\alpha, \mathbf{U}_\beta, 0 \right)$$

is presented and its relations are proved. Two theorems proved that the transformation $\mathbf{\Phi} = E_W \mathbf{Z}$ is completely canonical and is a solution of the Hamiltonian system (30). Recurrence relations are obtained. A simplified algorithm for evaluating the vectors $\mathbf{u}^{(n)}$ and $\mathbf{U}_\alpha^{(n)} + \mathbf{U}_\beta^{(n)}$ transformations and its inverse Kamel [3] is verified to apply in our case.

References

[1] Hori, G. (1966a). *Space Mathematics*. Part. 3, Amer. Math. Soc.

[2] Deprit, A., (1969). Canonical Transformation Depending on a Small Parameter. *Celest. Mech.*, *1*, 12-30

[3] Kamel, A. A. (1970). Perturbation method in the theory of nonlinear oscillations. *Celest. Mech. 3*, 90-106.

[4] Henrard, J. (1970). On a Perturbation Theory Using Lie Transforms. *Celest. Mech. 3*, 107-120.

[5] Hori, G., (1966b). Theory of General Perturbations with Unspecified Canonical Variables". Publications of the Astronomical Society of Japan. *18*(4), 287-296.

[6] Campbell, J. A. & Jefferys, W. H. (1970). Equivalence of Perturbation theories of Hori-Deprit. *Celest. Mech. 2*, 467-473, DOI: 10.1007/BF01625278

[7] Mersman, W. A. (1970b). Explicit Recursive Algorithm for Construction of Equivalent Canonical Transformations. *Celest. Mech. 3*, 81.

[8] Hori, G. (1971). Theory of General Perturbation for Non-Canonical Systems. *Publ. Astro. Soc. Japan, 23*, 567-587.

[9] Shniad, H. (1970). The equivalence of von Zeipel mappings and Lie transforms. *Celest. Mech. 2*, 114-120.

[10] Mersman, W. A. (1971). Explicit recursive algorithms for the construction of equivalent canonical transformations. *Celest. Mech. 3*, 384-389. DOI: 10.1007/BF01231807.

[11] Sessin, W. (1985). A note on the integration of the equations of Lie-Deprit's method for unspecified canonical variables. *Celest. Mech. 35*, 19-21.

[12] Varadi, F. (1985). Two-parameter Lie transforms. *Celest. Mech. 36*, 133-142.

[13] Ahmed, M. K. M. (1993). Multiple-Parameter Lie Transform. *Earth, Moon, and Planets*, *61*, 21-28, DOI: 10.1007/BF01229114.

[14] Cui, D. & Garfinkel, B. (1985). A variant of the Hori-Lie series method. *Celest. Mech.*, *35*, 89-94.

[15] Samko, S. G. Kilbas, A. A. & Marichev, O. I. (1993). Fractional Integrals and Derivatives-Theory and Applications. *Gordon and Breach, Longhorne, PA.*

[16] Kilbas, A. A. Srivastava, H. M. & Trujillo, J. J. (2006). *Theory and Applications of Fractional Differential Equations*, Amsterdam, Elsevier.

[17] Magin, R. L. (2006). *Fractional Calculus in Bioengineering*. Connecticut, Begell House Publisher, Inc.

[18] Podlubny, I. (1999). *Fractional Differential Equations. Mathematics in Science and Engineering Vol. 198*, San Diego, Academic Press.

[19] Miller, K. S. & Ross, B. (1993). *An Introduction to the Fractional Integrals and Derivatives-Theory and Applications*. John Wiley and Sons.

[20] Riewe, F. (1996). Nonconservative Lagrangian and Hamiltonian mechanics. *Phys. Rev. E*, *53*, 1890-1899.

[21] Riewe, F. (1997). Mechanics with fractional derivatives. *Phys. Rev. E*, *55*, 3581-3592.

[22] Agrawal, Om. P. (2002). Formulation of Euler-Lagrange equations for fractional variational problems. *J. Math. Anal. Appl.*, *272*, 368-379.

[23] Agrawal, Om. P. (2006). Fractional variational calculus and the transversality conditions. *J. Phys. A: Math. Gen.*, *39*, 10375-10384.

[24] Agrawal, Om. P. (2007). Fractional variational calculus in terms of Riesz fractional derivatives. *J. Phys. A: Math. Theor.*, *40*, 6287-6303.

[25] Baleanu, D. & Muslih, S. (2005a). Lagrangian formulation of classical fields within Riemann-Liouville fractional derivatives. *Phys. Scripta*, *72*(2-3), 119-121.

[26] Baleanu, D. & Muslih, S. (2005b). Formulation of Hamiltonian equations for fractional variational problems. *Czech. J. Phys.*, *55*(6), 633-642 .

[27] Baleanu, D. (2008). New Applications of Fractional Variational Principles. *Reports on Mathematical Physics*. *61*(2),

[28] Oldham, K. B. & Spanier, J. (1974). The Fractional Calculus, New York, Academic Press.

[29] Eqab, M. R. & Ababneh, B. S. (2008). Hamilton-Jacobi fractional mechanics. *J. Math. Anal. Appl.*, *344*, 799-805.

[30] Abd El-Salam, F. A. (2011). Construction of Some New Fractional Canonical Transformations and their Generating Functions. *Indian Journal of Science and Technology*. *5*(10), 3482-3499.