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Analytical approximate solutions of fractionel convection-diffusion equation with modified Riemann-Liouville derivative by means of fractional variational iteration method

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Abstract

In this article, an analytical approximate solution of nonlinear fractional convection-diffusion with modified Riemann-Liouville derivative was obtained with the help of fractional variational iteration method (FVIM). A new application of fractional variational iteration method (FVIM) was extended to derive analytical solutions in the form of a series for this equation. It is indicated that the solutions obtained by the FVIM are reliable and an effective method for strongly nonlinear partial equations with modified Riemann-Liouville derivative.

Keywords: Fractional variational iteration method; fractional convection-diffusion equation; Riemann-Liouville derivative

1. Introduction

In recent years, considerable interest in fractional calculus used in many fields such as electromagnetics, acoustics, viscoelasticity, electrochemistry, cosmology, viscoelasticity, diffusion, edge detection, turbulence, signal processing material science, physics and engineering are have been successfully modelled by linear or nonlinear fractional order differential equations [1-8]. Das at al. have obtained an approximate analytical solution of the fractional diffusion equation with absorbent term and external force [9], Fractional convection-diffusion equation with nonlinear source term by Momani and Yildirim [10], space-time fractional advection-dispersion equation by Yildirim and Kocak [11], fractional Zakharov-Kuznetsov equations by Yildirim and Gulkanat [12], integro-differential equation by El-Shahed [13], non-Newtonian flow by Siddiqui et al. [14], fractional PDEs in fluid mechanics by Yildirim [15], reaction-diffusion Brusselator system with fractional time derivative [16].

The homotopy perturbation method (HPM) [9-23], the Adomian decomposition method [23-30], homotopy analysis method (HAM) [31] and the variational iteration method (VIM), proposed by Ji-Huan He [32-44], was successfully applied to autonomous ordinary and partial differential equations and other fields. Ji-Huan He [37] was the first to apply the variational iteration method to fractional differential equations and a new modified Riemann-Liouville left derivative is suggested by Jumarie [45-49] and also Momani [30] used the Adomian decomposition method and HPM [10] for solving the nonlinear fractional convection-diffusion equation. Recently, Wu [50] researched local behaviors of the fractal dynamics using the local fractional derivative. Using of the fractional variational iteration method was given numerical simulations of differential equations and fractional differential equations [51]. In [52, 53], fractional nonlinear differential equations can be solved by the fractional variational iteration method. More recently, Merdan et al. [54] and [55], have studied the differential transform method for the approximate solutions of the Fornberg-Whitham equation and also Merdan [56], [57] and [58] have used the fractional variational iteration method for solving numerically the time fractional reaction-diffusion equation and Swift-Hohenberg (S–H) equation with modified Riemann-Liouville derivative.

In this paper, we extend the application of the VIM in order to derive analytical approximate solutions to nonlinear fractional convection-diffusion problem

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^2 u}{\partial x^2}-c\frac{\partial u}{\partial x}+\Phi(u)+f(x,t), \quad (1)$$
$$0<x\le 1,\quad 0<\alpha\le 1,\quad t>0,$$

$$u(x,0)=h(x),\quad 0<x\le 1, \quad (2)$$

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where $\Phi(u)$ is some reasonable nonlinear function of u which is chosen as a potential energy, c is a constant, α is a parameter describing the order of the time-fractional derivative. The fractional derivative is considered in the modified Riemann-Liouville derivative. $u(x,t)$ is assumed to be a causal function of time, i.e., vanishing for $t<0$. The convection–diffusion equations are widely used in science and engineering as mathematical models for computational simulations, such as in oil reservoir simulations, transport of mass and energy, and global weather production, in which an initially discontinuous profile is propagated by diffusion and convection, the latter with a speed of c.

The goal of this paper is to extend the application of the variational iteration method to solve fractional nonlinear convection-diffusion equations with modified Riemann-Liouville derivative.

This paper is organized as follows:
In section 2, we are given brief definitions related to the fractional calculus theory. In section 3, we define the solution procedure of the fractional variational iteration method to show efficiency of this method, we present the application of the FVIM for the fractional nonlinear convection-diffusion equations with modified Riemann-Liouville derivative and numerical results in Section 4. The conclusions are then given in the final section 5.

2. Basic definitions

Here, some basic definitions and properties of the fractional calculus theory which can be found in [1-8, 45-49] are given.

Definition 1. Let the special operator D^{α} that we choose to use, which requires the dependent variable f to be continuous and $\lceil \alpha \rceil$-times differentiable in the independent variable x, is defined by

$$D^{\alpha} f(x) = D^{\lceil \alpha \rceil} I^{\lceil \alpha \rceil - \alpha} f(x), \alpha \notin N, \tag{3}$$

which is called the Riemann-Lioville fractional derivative of order α.

$$D^{\alpha} f(x) = \frac{d^m}{dx^m} f(x), \alpha = m \in N, D^0 f(x) = f(x).$$

Definition 2. The left-sided Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_{\mu}, \mu \geq -1$ is defined as

$$J_a^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} f(\tau) d\tau,$$
$$\text{for } \alpha > 0, x > 0 \text{ and } J_a^0 f(x) = f(x). \tag{4}$$

The properties of the operator J^{α} can be found in [1, 2].

Definition 3. The modified Riemann-Liouville derivative[47-48] is defined as

$$_0D_{\alpha}^{x} f(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_x^0 (x-\tau)^{m-\alpha} \left(f(\tau) - f(0) \right) d\tau, \tag{5}$$

where $x \in [0,1], m-1 \leq \alpha < m$ and $m \geq 1$.

Definition 4. Fractional derivative of compounded functions [47-48] is defined as

$$d^{\alpha} f \cong \Gamma(1+\alpha) df, 0<\alpha<1 \tag{6}$$

Definition 5. The integral with respect to $(dx)^{\alpha}$ [47-48] is defined as the solution of the fractional differential equation

$$dy \cong f(x)(dx)^{\alpha}, \quad x \geq 0, \quad y(0)=0, \quad 0<\alpha<1 \tag{7}$$

Lemma 1. Let $f(x)$ denote a continuous function [47-48] then the solution of the Eq. (7) is defined as

$$y = \int_0^x f(\tau)(d\tau)^{\alpha} = \alpha \int_0^x (x-\tau)^{\alpha} f(\tau) d\tau, 0<\alpha<1. \tag{8}$$

For example, if $f(x) = x^{\beta}$ in Eq. (8) one obtains

$$\int_0^x \tau^{\beta} (d\tau)^{\alpha} = \frac{\Gamma(\beta+1)\Gamma(\alpha+1)}{\Gamma(\alpha+\beta+1)} x^{\beta+\alpha}, 0<\alpha<1 \tag{9}$$

Definition 6. Assume that the continuous function $f: R \to R, x \to f(x)$ has a fractional derivative of order $k\alpha$, for any positive integer k and any α, $0<\alpha \leq 1$; then the following equality holds, which is

$$f(x+h) = \sum_{k=0}^{\infty} \frac{h^{\alpha k}}{\alpha k!} f^{\alpha k}(x), \quad 0<\alpha \leq 1. \tag{10}$$

On making the substitution $h \to x$ and $x \to 0$ we obtain the fractional Mc-Laurin series

$$f(x) = \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{\alpha k!} f^{\alpha k}(0), \quad 0<\alpha \leq 1. \tag{11}$$

3. Fractional variational iteration method

To describe the solution procedure of the fractional variational iteration method, we consider the following fractional differential equation [50-58]:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} - c\frac{\partial u}{\partial x} + \Phi(u) + f(x,t), \quad 0 < x \le 1, \quad 0 < \alpha \le 1, \quad t > 0, \tag{12}$$

According to the VIM, we can build a correct functional for Eq. (12) as follows

$$u_{n+1}(x,t) = u_n(x,t) + I^\alpha\left[\lambda(x,t)\left(\frac{\partial^\alpha u_n(x,t)}{\partial t^\alpha} - \left(\frac{\partial^2 u_n(x,t)}{\partial x^2} - c\frac{\partial u_n(x,t)}{\partial x} + \Phi(u_n(x,t)) + f(x,t)\right)\right)\right]$$

$$u_{n+1}(x,t) = u_n(x,t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\lambda(x,\tau)\left(\frac{\partial^\alpha u_n(x,\tau)}{\partial \tau^\alpha} - \left(\frac{\partial^2 u_n(x,\tau)}{\partial x^2} - c\frac{\partial u_n(x,\tau)}{\partial x} + \Phi(u_n(x,\tau)) + f(x,\tau)\right)\right)d\tau \tag{13}$$

Using Eq. (5), we obtain a new correction functional

$$u_{n+1}(x,t) = u_n(x,t) + \frac{1}{\Gamma(\alpha+1)}\int_0^t \lambda(x,\tau)\left(\frac{\partial^\alpha u_n(x,\tau)}{\partial \tau^\alpha} - \left(\frac{\partial^2 u_n(x,\tau)}{\partial x^2} - c\frac{\partial u_n(x,\tau)}{\partial x} + \Phi(u_n(x,\tau)) + f(x,\tau)\right)\right)(d\tau)^\alpha \tag{14}$$

It is obvious that the sequential approximations $u_k, k \ge 0$ can be established by determining λ, a general Lagrange's multiplier which can be identified optimally with the variational theory. The function $\tilde{u}_n$ is a restricted variation which means $\delta\tilde{u}_n = 0$. Therefore, we first designate the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximations $u_{n+1}(x,t), n \ge 0$ of the solution $u(x,t)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0. The initial values are usually used for choosing the zeroth approximation u_0. With λ determined, then several approximations $u_k, k \ge 0$ follows immediately [51, 52, 58]. Consequently, the exact solution may be procured by using

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) \tag{15}$$

4. Applications

In this section, we present the solution of nonlinear fractional partial differential equations as the applicability of FVIM.

Examples 4.1. Consider the nonlinear fractional convection-diffusion equation where $0 < \alpha \le 1, c = 1, 0 < x \le 1, t > 0,$

and $\Phi(u) = u\frac{\partial^2 u}{\partial x^2} - u^2 + u$, we get

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} + u\frac{\partial^2 u}{\partial x^2} - u^2 + u \tag{16}$$

with initial conditions

$$u(x,0) = e^x, [10,30]. \tag{17}$$

Construct the following functional:

$$u_{n+1}(x,t) = u_n(x,t) + \frac{1}{\Gamma(\alpha+1)}\int_0^t \lambda(x,\tau)\left\{\frac{\partial^\alpha u_n(x,\tau)}{\partial \tau^\alpha} - \frac{\partial^2 u_n(x,\tau)}{\partial x^2} + \frac{\partial u_n(x,\tau)}{\partial x} - u_n(x,\tau)\frac{\partial^2 u_n(x,\tau)}{\partial x^2} + u_n^2(x,\tau) - u_n(x,\tau)\right\}(d\tau)^\alpha. \tag{18}$$

We have

$$\delta u_{n+1}\left(x,t\right)=\delta u_n(x,t)+\frac{1}{\Gamma\left(\alpha+1\right)}\delta\int_0^t\lambda\left(x,\tau\right)\begin{Bmatrix}\frac{\partial^\alpha u_n(x,\tau)}{\partial\tau^\alpha}-\frac{\partial^2 u_n(x,\tau)}{\partial x^2}+\frac{\partial u_n(x,\tau)}{\partial x}\\-u_n(x,\tau)\frac{\partial^2 u_n(x,\tau)}{\partial x^2}+u_n^2(x,\tau)-u_n(x,\tau)\end{Bmatrix}\left(d\tau\right)^\alpha$$
$$=\delta u_n+\lambda\delta u_n\big|_{\tau=t}-\frac{1}{\Gamma\left(\alpha+1\right)}\int_0^t\frac{\partial^\alpha\lambda(x,\tau)}{\partial\tau^\alpha}\delta u_n(x,\tau)\left(d\tau\right)^\alpha \tag{19}$$

Similarly, we can get the coefficients of δu_n to zero:

$$1+\lambda\left(x,\tau\right)\big|_{\tau=t}=0,\quad \frac{\partial^\alpha\lambda(x,\tau)}{\partial\tau^\alpha}=0\,. \tag{20}$$

The generalized Lagrange multiplier can be identified by the above equations,

$$\lambda\left(x,t\right)=-1. \tag{21}$$

Substituting Eq. (21) into Eq. (18) produces the iteration formulation as follows

$$u_{n+1}\left(x,t\right)=u_n(x,t)-\frac{1}{\Gamma\left(\alpha+1\right)}\int_0^t\begin{Bmatrix}\frac{\partial^\alpha u_n(x,\tau)}{\partial\tau^\alpha}-\frac{\partial^2 u_n(x,\tau)}{\partial x^2}+\frac{\partial u_n(x,\tau)}{\partial x}\\-u_n(x,\tau)\frac{\partial^2 u_n(x,\tau)}{\partial x^2}+u_n^2(x,\tau)-u_n(x,\tau)\end{Bmatrix}\left(d\tau\right)^\alpha \tag{22}$$

Taking the initial value $u_0\left(x,t\right)=u_0\left(x,0\right)=e^x$, we can derive

$$\begin{aligned}u_1\left(x,t\right)&=u_0(x,t)-\frac{1}{\Gamma\left(\alpha+1\right)}\int_0^t\begin{Bmatrix}\frac{\partial^\alpha u_0(x,\tau)}{\partial\tau^\alpha}-\frac{\partial^2 u_0(x,\tau)}{\partial x^2}+\frac{\partial u_0(x,\tau)}{\partial x}\\-u_0(x,\tau)\frac{\partial^2 u_0(x,\tau)}{\partial x^2}+u_0^2(x,\tau)-u_0(x,\tau)\end{Bmatrix}\left(d\tau\right)^\alpha\\&=e^x+\frac{e^xt^\alpha}{\Gamma\left(\alpha+1\right)},\\u_2\left(x,t\right)&=e^x+\frac{e^xt^\alpha}{\Gamma\left(\alpha+1\right)}+\frac{e^xt^{2\alpha}}{\Gamma\left(2\alpha+1\right)},\\u_3\left(x,t\right)&=e^x+\frac{e^xt^\alpha}{\Gamma\left(\alpha+1\right)}+\frac{e^xt^{2\alpha}}{\Gamma\left(2\alpha+1\right)}+\frac{e^xt^{3\alpha}}{\Gamma\left(3\alpha+1\right)},\\&\vdots\\u_n\left(x,t\right)&=e^x+\frac{e^xt^\alpha}{\Gamma\left(\alpha+1\right)}+\frac{e^xt^{2\alpha}}{\Gamma\left(2\alpha+1\right)}+\frac{e^xt^{3\alpha}}{\Gamma\left(3\alpha+1\right)}+\cdots+\frac{e^xt^{n\alpha}}{\Gamma\left(n\alpha+1\right)}\end{aligned} \tag{23}$$

Then, the approximate solutions in a series form are

$$\begin{aligned}u(x,t)&=\underset{n\to\infty}{Lim}\,u_n\left(x,t\right)=e^x\left(1+\frac{t^\alpha}{\Gamma\left(\alpha+1\right)}+\frac{t^{2\alpha}}{\Gamma\left(2\alpha+1\right)}+\frac{t^{3\alpha}}{\Gamma\left(3\alpha+1\right)}+\cdots\right)\\&=e^x\sum_{k=0}^{\infty}\frac{\left(t^\alpha\right)^k}{\Gamma\left(k\alpha+1\right)},\end{aligned} \tag{24}$$

which has the exact solution

$$u(x,t)=e^xE_\alpha\left(t^\alpha\right), \tag{25}$$

For the special case, $\alpha=1$ is

$$u(x,t)=e^x\sum_{k=0}^{\infty}\frac{\left(t^k\right)}{\Gamma\left(k+1\right)}=e^{x+t} \tag{26}$$

which is an exact solution to the nonlinear convection-diffusion equation.

Figure 1 is plotted for approximate solution of generalized time-fractional convection-diffusion equation found in Example 4.1.

Figure 2 is for approximate solution of Eq. (16) for $\alpha = 0.9, 0.8, 0.7, 0.6$.

Figures 3 and 4 are prepared to show the influence of a on the function $u(x,t)$. Figures 3 and 4 indicate that a decrease in the fractional order α by selecting the fixed $x = 0.5$ corresponds to an increase in the function. From Figs. 3 and 4 five sequential values of $\alpha = 1, 0.9, 0.8, 0.7, 0.6$ are seen.

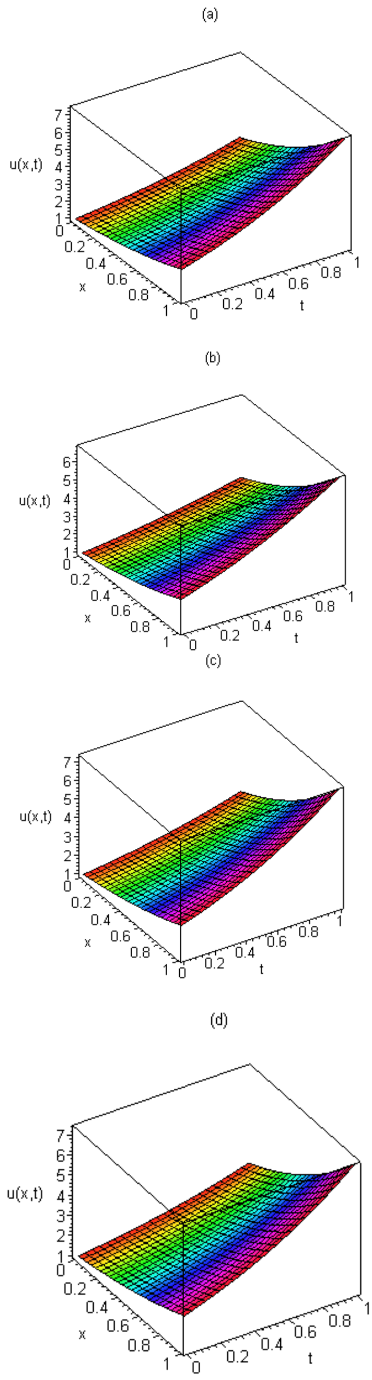


Fig. 1. The surface indicates the solution $u(x,t)$ for Eq. (16) when $\alpha = 1$. (a) Exact solution (b) $u_2(x,t)$-approximate solution, (c) $u_3(x,t)$-approximate solution and (d) $u_4(x,t)$-approximate solution

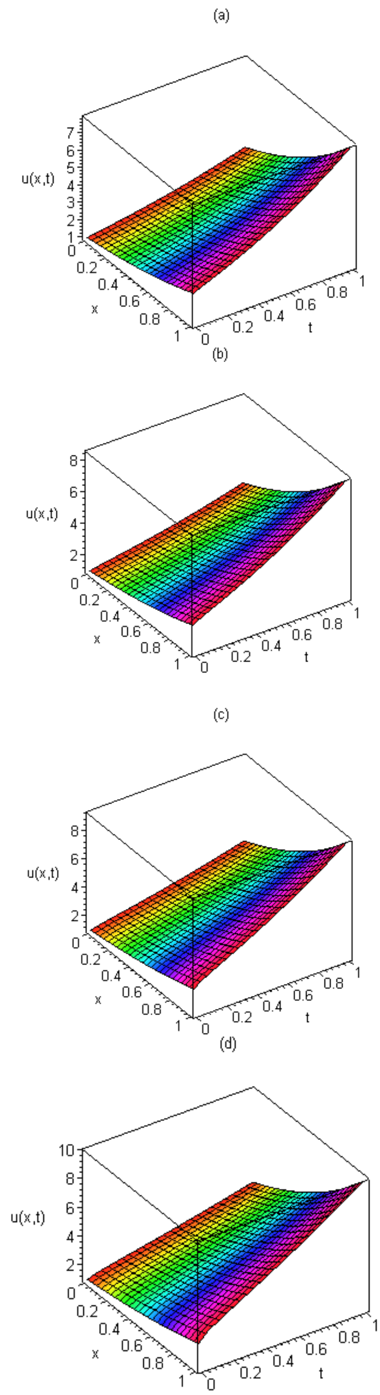


Fig. 2. The surface indicates the solution $u(x,t)$ for Eq. (16) (a) approximate solution $u_3(x,t)$ for $\alpha = 0.9$ (b) approximate solution $u_3(x,t)$ for $\alpha = 0.8$ (c) approximate solution $u_3(x,t)$ for $\alpha = 0.7$ and (d) approximate solution $u_3(x,t)$ for $\alpha = 0.6$

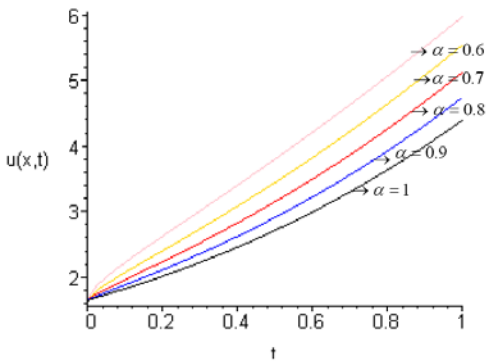


Fig. 3. Approx. solution $u_3(x,t)$ for $x = 0.5$

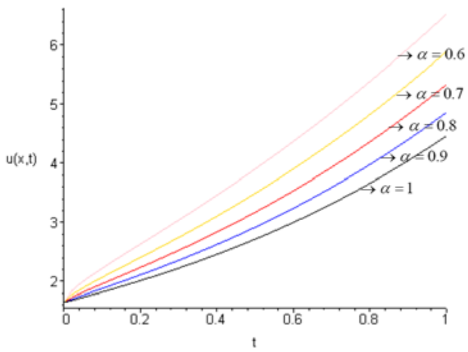


Fig. 4. Approx. solution $u_4(x,t)$ for $x = 0.5$

Eq. (16) is solved in [30] using the Adomian decomposition method and HPM [10]. FVIM solutions indicate that the present algorithm performs with considerable efficiency, simplicity and reliability. The results obtained from FVIM are fully compatible with the Adomian decomposition method and HPM.

Examples 4.2. We next consider the non-homogenous nonlinear fractional convection-diffusion equation where $0 < \alpha \le 1, c = 1, 0 < x \le 1, t > 0,$

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} - c\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial u}{\partial x} + u\frac{\partial^2 u}{\partial x \partial t} - 2x \qquad (27)$$

with initial conditions

$$u(x,0) = x^2. \qquad (28)$$

Construct the following functional:

$$u_{n+1}(x,t) = u_n(x,t) + \frac{1}{\Gamma(\alpha+1)}\int_0^t \lambda(x,\tau)\left\{\begin{array}{l}\frac{\partial^\alpha u_n(x,\tau)}{\partial \tau^\alpha} - \frac{\partial^2 u_n(x,\tau)}{\partial x^2} + \frac{\partial u_n(x,\tau)}{\partial x} \\ -\frac{\partial u_n(x,\tau)}{\partial \tau}\frac{\partial u_n(x,\tau)}{\partial x} - u_n(x,\tau)\frac{\partial^2 u_n(x,\tau)}{\partial \tau \partial x} + 2x\end{array}\right\}(d\tau)^\alpha \qquad (29)$$

we have

$$\begin{aligned}\delta u_{n+1}(x,t) &= \delta u_n(x,t) + \frac{1}{\Gamma(\alpha+1)}\delta\int_0^t \lambda(x,\tau)\left\{\begin{array}{l}\frac{\partial^\alpha u_n(x,\tau)}{\partial \tau^\alpha} - \frac{\partial^2 u_n(x,\tau)}{\partial x^2} + \frac{\partial u_n(x,\tau)}{\partial x} \\ -\frac{\partial u_n(x,\tau)}{\partial \tau}\frac{\partial u_n(x,\tau)}{\partial x} - u_n(x,\tau)\frac{\partial^2 u_n(x,\tau)}{\partial \tau \partial x} + 2x\end{array}\right\}(d\tau)^\alpha \\ &= \delta u_n + \lambda \delta u_n\big|_{\tau=t} - \frac{1}{\Gamma(\alpha+1)}\int_0^t \frac{\partial^\alpha \lambda(x,\tau)}{\partial \tau^\beta}\delta u_n(x,\tau)(d\tau)^\alpha\end{aligned} \qquad (30)$$

similarly, we can get the coefficients of δu_n to zero:

$$1 + \lambda(x,\tau)\big|_{\tau=t} = 0, \quad \frac{\partial^\alpha \lambda(x,\tau)}{\partial \tau^\alpha} = 0. \qquad (31)$$

The generalized Lagrange multiplier can be identified by the above equations,

$$\lambda(x,t) = -1. \qquad (32)$$

Substituting Eq. (32) into Eq. (29) produces the iteration formulation as follows

$$u_{n+1}(x,t)=u_n(x,t)-\frac{1}{\Gamma(\alpha+1)}\int_0^t\left\{\begin{array}{l}\frac{\partial^\alpha u_n(x,\tau)}{\partial\tau^\alpha}-\frac{\partial^2 u_n(x,\tau)}{\partial x^2}+\frac{\partial u_n(x,\tau)}{\partial x}\\ -\frac{\partial u_n(x,\tau)}{\partial\tau}\frac{\partial u_n(x,\tau)}{\partial x}-u_n(x,\tau)\frac{\partial^2 u_n(x,\tau)}{\partial\tau\partial x}+2x\end{array}\right\}(d\tau)^\alpha \tag{33}$$

Taking the initial value $u_0(x,t)=u_0(x,0)=x^2$, we can derive

$$\begin{aligned}u_1(x,t)&=u_0(x,t)-\frac{1}{\Gamma(\alpha+1)}\int_0^t\left\{\begin{array}{l}\frac{\partial^\alpha u_0(x,\tau)}{\partial\tau^\alpha}-\frac{\partial^2 u_0(x,\tau)}{\partial x^2}+\frac{\partial u_0(x,\tau)}{\partial x}\\ -\frac{\partial u_0(x,\tau)}{\partial\tau}\frac{\partial u_0(x,\tau)}{\partial x}-u_0(x,\tau)\frac{\partial^2 u_0(x,\tau)}{\partial\tau\partial x}+2x\end{array}\right\}(d\tau)^\alpha\\ &=x^2+\frac{(2-4x)t^\alpha}{\Gamma(\alpha+1)},\\ u_2(x,t)&=x^2+\frac{(2-4x)t^\alpha}{\Gamma(\alpha+1)}+\frac{4t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{(-12x^2+4x)\alpha\Gamma(\alpha)t^{2\alpha-1}}{\Gamma(\alpha+1)\Gamma(2\alpha)}+\frac{(32x-16)\alpha\Gamma(2\alpha)t^{3\alpha-1}}{\Gamma(\alpha+1)^2\Gamma(3\alpha)}.\end{aligned} \tag{34}$$

$$\begin{aligned}u(x,t)&=\underset{n\to\infty}{Lim}\,u_n(x,t)=x^2+\frac{(2-4x)t^\alpha}{\Gamma(\alpha+1)}\\ &+\frac{4t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{(-12x^2+4x)\alpha\Gamma(\alpha)t^{2\alpha-1}}{\Gamma(\alpha+1)\Gamma(2\alpha)}+\frac{(32x-16)\alpha\Gamma(2\alpha)t^{3\alpha-1}}{\Gamma(\alpha+1)^2\Gamma(3\alpha)}+\cdots,\end{aligned} \tag{35}$$

For the special case $\alpha=1$ is [58]

$$\begin{aligned}u(x,t)&=\underset{n\to\infty}{Lim}\,u_n(x,t)=x^2+2t-4tx\\ &+2t^2+4tx-12x^2t+16xt^2-8t^2+\cdots,\end{aligned} \tag{36}$$

Canceling the noise terms and keeping the non-noise terms yields the exact solution of Eq. (27) given by

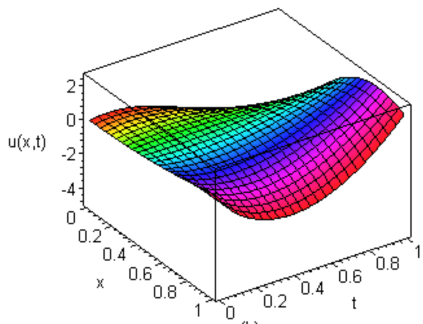
$$u(x,t)=x^2+2t \tag{37}$$

which is easily confirmed. This is formally proved right in [19].

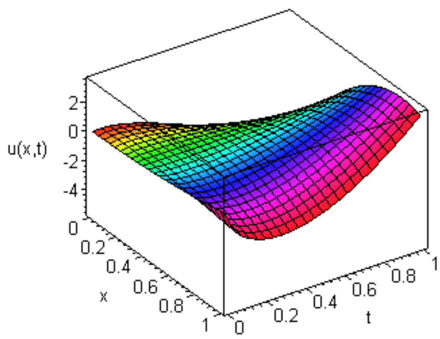
Finally, the solution surfaces of the non-homogenous nonlinear fractional convection-diffusion equation are depicted in Fig. 5 for different values of α. Figures 5 and 6 are prepared to show the influence of α on the function $u(x,t)$. It is clearly seen that a $u(x,t)$ increase with the increases in t for $\alpha=1, 0.9, 0.8, 0.7, 0.6$.

Eq. (27) is solved in [30] using the Adomian decomposition method and HPM[10]and the results in Fig. 5 compare well with those obtained from the Adomian decomposition method and HPM.

(a)



(b)



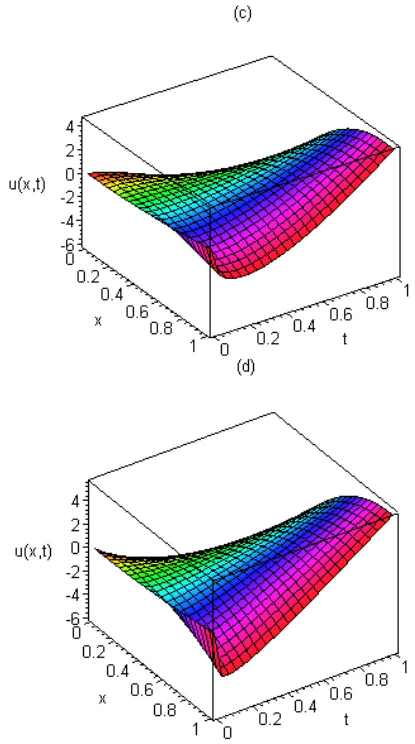


Fig. 5. The surface indicates the solution $u(x,t)$ for Eq. (27)**.** (a) approximate solution $u_2(x,t)$ for $\alpha = 0.9$ (b) approximate solution $u_2(x,t)$ for $\alpha = 0.8$ (c) approximate solution $u_2(x,t)$ for $\alpha = 0.7$ and (d) approximate solution $u_2(x,t)$ for $\alpha = 0.6$

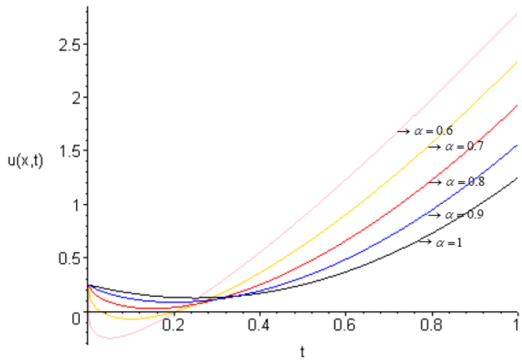


Fig. 6. Approx. solution $u_2(x,t)$ for $x = 0.5$

5. Conclusions

Variational iteration method is known as very powerful and an effective method for solving nonlinear problems and ordinary, partial, fractional, integral equations. In this paper, we have discussed modified variational iteration method having integral w. r. t. $(d\tau)^{\alpha}$ used for the first time by Jumarie. The obtained results indicate that this method is powerful and meaningful for solving the nonlinear fractional differential equations. Two examples indicate that the results of variational iteration method having integral w. r. t. $(d\tau)^{\alpha}$ are in excellent agreement with those obtained by classical HPM [10] and Adomian decomposition method [30] which is available in the literature.

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