

Geometrical Categories of Generalized Lie Groups and Lie Group-Groupoids

M. R. Farhangdoost* and T. Nasirzade

Department of Mathematics, College of Sciences, Shiraz University, P.O. Box 71457- 44776, Shiraz, Iran
E-mail: Farhang@shirazu.ac.ir

Abstract

In this paper we construct the category of coverings of fundamental generalized Lie group-groupoid associated with a connected generalized Lie group. We show that this category is equivalent to the category of coverings of a connected generalized Lie group. In addition, we prove the category of coverings of generalized Lie group-groupoid and the category of actions of this generalized Lie group-groupoid on a connected generalized Lie group are equivalent.

Keywords: Lie groupoid; Lie group; Generalized Lie group; Homotopy of topological groups

1. Introduction

Certain manifolds such as the torus T^2 also have in multiplication structure, a generalized group structure; moreover, the generalized group operations are C^∞ . Manifold such as these are called generalized Lie group or top space.

Generalized Lie group is an important class of manifolds. In this generalized setting, several authors (Molaei M. R., Mehrabi M., Oloomi A., Araujo J., Tahmoresi A., etc.) studied various aspects and concepts of generalized groups and generalized Lie groups (top spaces) [1]. This structure is a different structure from the useful and remarkable Lie groupoid which has been introduced and studied by Gheorghe Ivan [2].

Precisely, the concept of generalized Lie groups is defined by a set of definitions, i.e.:

Definition 1. ([3, 1]) A non-empty Hausdorff smooth d -dimensional manifold T is called a generalized Lie group, if there is a product m on T such that $m(t,s) \in T$, for every $t, s \in T$ and satisfied in the following conditions:

1. $m(m(r,s),t) = m(r,m(s,t))$, for all $r, s, t, \in T$;
2. For each $t \in T$, there is a unique $e(t) \in T$ such that $m(t,e(t)) = m(e(t),t) = t$;
3. For all $t, s \in T$, $e(m(t,s)) = m(e(t),e(s))$;
4. For each $t \in T$, there is $s \in T$ such that $m(t,s) = m(s,t) = e(t)$;
 s is called the inverse of t and denoted by t^{-1} .
5. The following mappings are smooth.

$$m : T \times T \rightarrow T \quad \text{and} \quad m_1 : T \rightarrow T \\ (t, s) \rightarrow m(t, s) \quad \quad \quad t \rightarrow t^{-1}$$

Example 2. Each Lie group is a generalized Lie group. [3]

Example 3. If T is a generalized Lie group, then $T \times T$ is a generalized Lie group with action:

$$m((a,b), (c,d)) = (m(a,c), m(b,d));$$

also, we have:

$$e(a,b) = (e(a), e(b)) \\ (a,b)^{-1} = (a^{-1}, b^{-1}).$$

Another mathematical object which is used in this paper is Lie groupoid [2, 4]. A groupoid is a category in which every arrow is invertible. More precisely, a groupoid consists of two sets G and G_0 called the set of morphisms or arrows and the set of objects of groupoid respectively, together with two maps $\alpha, \beta: G \rightarrow G_0$ called source and target maps respectively, a map $1_0 : G_0 \rightarrow G$, $x \rightarrow x_0$ called the object map, an inverse map $i : G \rightarrow G$, $a \rightarrow a^{-1}$ and a composition $G_2 = G \times_{\alpha} G \times_{\beta} G \rightarrow G$, $(b,a) \rightarrow (boa)$ defined on the pullback set:

$$G \times_{\alpha} G \times_{\beta} G = \{(b,a) \in G \times G \mid \alpha(b) = \beta(a)\},$$

these maps should satisfy the following conditions:

1. $\alpha(boa) = \alpha(a)$ and $\beta(boa) = \beta(b)$ for all $(b,a) \in G_2$;
2. $co(boa) = (cob)oa$ such that $\alpha(b) = \beta(a)$ and $\alpha(c) = \beta(b)$, for all $a, b, c \in G$;

*Corresponding author

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3. $\alpha(1_x) = \beta(1_x) = x$, for all $x \in G_0$;
4. $\alpha o 1_{\alpha(a)} = a$ and $1_{\beta(a)} o \alpha = a$, for all $a \in G$;
5. $\alpha(a^{-1}) = \beta(a)$ and $\beta(a^{-1}) = \alpha(a)$, for all $a \in G$. [2]

Definition 4. A groupoid G over G_0 is called Lie groupoid if G and G_0 are manifolds, α and β are surjective submersions and the composition is smooth map. [4, 5]

Note: In this paper by $\pi_1 T$ we mean the set of all homotopy classes of paths in T , where T is a manifold.

In the second section of this paper we show that, for generalized Lie group T , the fundamental group $\pi_1 T$ is a generalized Lie group. Also, by definition of generalized Lie group-groupoid, we show that $\pi_1 T$ is a generalized Lie group-groupoid. Then we find a functor from category of generalized Lie groups to category of generalized Lie group-groupoids.

Section three is devoted to study a category $GLgGdCov(T)$ of the coverings of generalized Lie group T and a category $GLgGdOp(T)$ of the actions of T on generalized Lie groups. We show that these two categories are equivalent.

2. On Generalized Lie Group and Lie Group-Groupoids

We begin this section by a connective proposition:

Proposition 5. If T is a generalized Lie group, then $\pi_1 T$ is a generalized Lie group.

Proof: If (T, m) be a generalized Lie group, we show that $(\pi_1 T, \pi_1 m)$ is a generalized Lie group, where we define

$$\begin{aligned} \pi_1 m : \pi_1 T \times \pi_1 T &\rightarrow \pi_1 T \\ ([f], [g]) &\rightarrow [m(f, g)] = [h]; \end{aligned}$$

where $h(i) = m(f(i), g(i))$.

Also, we have:

$$e([f]) = [e(f)]$$

and

$$[f]^{-1} = [f^{-1}].$$

Then it is easy to show $(\pi_1 T, \pi_1 m)$ is generalized Lie group.

Now, we join two different important objects of geometry, i.e. we define generalized Lie group-groupoid.

Definition 6. A generalized Lie group-groupoid or $GLG - Gd$ is a Lie groupoid endowed with a structure of generalized Lie group such that the action $m : T \times T \rightarrow T$, the unit map $e : T \rightarrow T$ and inverse map, which are the structure maps of generalized Lie group, are Lie groupoid morphism. Moreover, there exists an interchange law

$$m((boa), (doc)) = (m(b, d))o(m(a, c)).$$

Example 7. If T is a generalized Lie group, then $T \times T$ is a $GLG - Gd$, because: since T is a manifold, $T \times T$ is a Lie groupoid [6], and by the Example 3, $T \times T$ is a generalized Lie group. Now we show that the generalized Lie group structure maps of $T \times T$ are groupoid morphisms. Suppose that the action of T is m , and the action of $T \times T$ is m_1 . We have

$$\begin{aligned} m_1 \left(((a, b), (a', b')) o ((c, d), (c', d')) \right) & \\ = m_1 \left((a, b) o (c, d), (a', b') o (c', d') \right) & \\ = m_1 \left((a, d), (a', d') \right) & \\ = (m(a, a'), m(d, d')) & \\ = (m(a, a'), m(b, b')) o (m(c, c'), m(d, d')) & \\ = m_1 \left((a, b), (a', b') \right) o m_1 \left((c, d), (c', d') \right) & \end{aligned}$$

and

$$\begin{aligned} (\alpha o m_1) \left((a, b) o (c, d) \right) & \\ = \alpha \left(m(a, c), m(b, d) \right) & \\ = m(b, d) & \\ = (m o \alpha') \left((a, b) o (c, d) \right). & \end{aligned}$$

Also, $\beta o m_1 = m o \beta$.

Similarly, it can be shown that the unit map and the inverse map of the generalized Lie group are groupoid morphisms. Moreover, we have

$$\begin{aligned} m_1 \left((b, a) o (a, c), (b', a') o (a', c') \right) & \\ = m_1 \left((b, c), (b', c') \right) = (m(b, b'), m(c, c')) & \\ = (m(b, b'), m(a, a')) o (m(a, a'), m(c, c')) & \\ = m_1 \left((b, a), (b', a') \right) o m_1 \left((a, c), (a', c') \right). & \end{aligned}$$

Then $T \times T$ is a $GLG - Gd$.

Proposition 8. If (T, m) is a generalized Lie group, then $(\pi_1 T, \pi_1 m)$ is a $GLG - Gd$.

Proof: By the Proposition 5, $\pi_1 T$ is a generalized Lie group. From [4] we know that the $\pi_1 m$ is a morphism of Lie groupoids. Also, we have:

$$\begin{aligned} & \pi_1 m(([b]o[a]), ([d]o[c])) \\ &= \pi_1 m([boa], [doc]) \\ &= [m((boa), (doc))] = [(m(b, d))o(m(a, c))] \\ &= [m(b, d)]o[m(a, c)] \\ &= (\pi_1 m([b], [d]))o(\pi_1 m([a], [c])). \end{aligned}$$

Now, we present the concept of the morphism in this new category:

Definition 9. Let T_1 and T_2 be two generalized Lie group-groupoids. A morphism $f: T_1 \rightarrow T_2$ of generalized Lie group-groupoids is a morphism of the underlying Lie groupoids, which preserves the generalized Lie group structure, i.e. $f(m_1(a, b)) = m_2(f(a), f(b))$, where m_1 and m_2 are the actions T_1 and T_2 , respectively. [4]

Proposition 10. There exists a functor from the category GLg of generalized Lie groups to the category $GLG - Gd$ of generalized Lie group-groupoids.

Proof: Let T be a generalized Lie group. Then, from Example 7, $T \times T$ is a $GLG - Gd$. If $f: T_1 \rightarrow T_2$ is a morphism of generalized Lie groups, then

$$\Gamma(f) : T_1 \times T_1 \rightarrow T_2 \times T_2 \\ (a, b) \rightarrow (f(a), f(b))$$

is a morphism of $GLG - Gds$. So, Γ is a functor from the category GLg to the category $GLG - Gd$.

3. Coverings and Actions of Generalized Lie Group-Groupoids

We start this section by definition of covering morphism of Lie groupoids.

Definition 11. Let $p : \tilde{G} \rightarrow G$ be a morphism of Lie groupoids. For each $\tilde{x} \in \tilde{G}_0$, if the restriction $\tilde{G}_{\tilde{x}}$ of p is a diffeomorphism, p is called the covering morphism of Lie groupoids.

From [4], we know that the morphism $p : \tilde{G} \rightarrow G$ is covering morphism of Lie groupoids iff the morphism

$$(p, \alpha) : \tilde{G} \rightarrow G \times_{p_0} \tilde{G}_0$$

is a diffeomorphism.

Definition 12. A morphism $f: T_1 \rightarrow T_2$ of generalized Lie group-groupoids is called a covering morphism of generalized Lie group-groupoids, if it is a covering morphism of underlying Lie groupoids.

Proposition 13. Let (T_1, m_1) and (T_2, m_2) be two connected generalized Lie groups and $p: T_1 \rightarrow T_2$

be a covering morphism of generalized Lie groups. Then the morphism $\pi_1 p : \pi_1 T_1 \rightarrow \pi_1 T_2$ is a covering morphism of generalized Lie group-groupoids.

Proof: From [6], $\pi_1 p : \pi_1 T_1 \rightarrow \pi_1 T_2$ is a covering morphism of Lie groupoids. Then we show that $\pi_1 p$ preserve the generalized Lie group structure:

$$\begin{aligned} & \pi_1 p (\pi_1 m([a], [b])) \\ &= \pi_1 p([m_1(a, b)]) \\ &= [p m_1(a, b)] \\ &= [m_2((p o a), (p o b))] \\ &= \pi_1 m_2([p o a], [p o b]) \\ &= \pi_1 m_2(\pi_1 p[a], \pi_1 p[b]). \end{aligned}$$

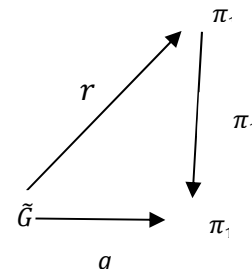
Let T be a connected generalized Lie group. $GLgCov(T)$ is a category whose objects are smooth covering maps $p : \tilde{T} \rightarrow T$ of generalized Lie groups and a morphism from $p : \tilde{M} \rightarrow M$ to $q : \tilde{N} \rightarrow N$ is a map $r : \tilde{M} \rightarrow \tilde{N}$ satisfying the condition $p = q o r$.

Also, $GLgGdCov(\pi_1 T)$ is a category with smooth covering morphisms $p : \tilde{G} \rightarrow \pi_1 T$ of generalized Lie group-groupoid as objects, where \tilde{G}_0 is connected generalized Lie group. A morphism from $p : \tilde{G} \rightarrow \pi_1 T$ to $q : \tilde{H} \rightarrow \pi_1 T$ is a map $r : \tilde{G} \rightarrow \tilde{H}$ satisfying the condition $p = q o r$.

Now let us state a proposition from [4] to be necessary for the proof of the following proposition.

Proposition 14. Let T be a connected generalized Lie group and $q : \tilde{G} \rightarrow \pi_1 T$ be covering morphism of generalized Lie group-groupoids. Let $\tilde{T} = \tilde{G}_0$ and $p = q_0 : \tilde{T} \rightarrow T$. Let A denote an atlas consisting of the liftable charts. Then the smooth structure over $\pi_1 T$ is the unique structure such that the following hold:

1. $p : \tilde{T} \rightarrow T$ is a covering map;
2. There exists an isomorphism $r : \tilde{G} \rightarrow \pi_1 \tilde{T}$ which is the identical on objects such that the following diagram is commutative:



Here, we present two crucial theorems which equal two different categories.

Theorem 15. Let T be a connected generalized Lie group. Then the categories $GLgCov(T)$ and $GLgGdCov(\pi_1 T)$ are equivalent.

Proof: Firstly, let us define functor $\Gamma: GLgCov(T) \rightarrow GLgGdCov(\pi_1 T)$ in the following way:

Let $p: \tilde{T} \rightarrow T$ be a covering map of generalized Lie groups. Then by the Proposition 13, $\pi_1 p: \pi_1 \tilde{T} \rightarrow \pi_1 T$ is a covering morphism of generalized Lie group-groupoids. That is, $\Gamma(p) = \pi_1 p$ is a covering morphism of generalized Lie group-groupoids.

Secondly, let us define a functor $\varphi: GLgGdCov(\pi_1 T) \rightarrow GLgCov(T)$ as follows:

Let $q: \tilde{G} \rightarrow \pi_1 T$ be covering morphism of generalized Lie group-groupoids, where T and \tilde{G}_0 are connected generalized Lie groups. By the Proposition 14, there exists a liftable manifold structure on \tilde{T} such that $p = q_0: \tilde{T} \rightarrow T$ is the covering map of the generalized Lie groups. Thus $\varphi(q) = q_0 = p$ is a covering map of the generalized Lie groups.

It is easy to show that natural equivalences $\Gamma\varphi \simeq 1_{GLgGdCov(\pi_1 T)}$ and $\varphi\Gamma \simeq 1_{GLgCov(T)}$ are satisfied.

Definition 16. Let T be a $GLG - Gd$ and M be a generalized Lie group. An action of T on M via generalized Lie group homomorphism $w: M \rightarrow T_0$ is a smooth map $\lambda: T_\alpha \times_w M \rightarrow M$, $(t, n) \mapsto t.n$ satisfying the conditions:

1. $w(t.n) = \beta(t)$;
2. $t_1.(t_2.n) = (t_1 o t_2).n$;
3. $(1_{w(n)}) . n = n$;
4. $(t_1.n_1) \times (t_2.n_2) = (t_1 \times t_2).(n_1 \times n_2)$.

Remark 1. Let $p: \tilde{T} \rightarrow T$ be covering morphism of $GLG - Gds$. We take $M = \tilde{T}_0$ and $w = p_0: \tilde{T}_0 \rightarrow T_0$. Thus we obtain an action $\lambda: T_\alpha \times_w \tilde{T}_0 \rightarrow \tilde{T}_0$, $(a, \tilde{x}) \mapsto a.\tilde{x} = \tilde{\beta}(\tilde{a})$ of T on $M = \tilde{T}_0$ by $w = p_0$, where \tilde{a} is lifting of a .

Remark 2. Let T be a $GLG - Gd$ acting on generalized Lie group M via generalized Lie group homomorphism $w: M \rightarrow T_0$. Then we have action generalized Lie group $T * M$ whose objects manifold is generalized Lie group M . Future, action generalized Lie group $T * M$ is a $GLG - Gd$ by the operation $(a, x).(b, y) = (a.b, x.y)$, where the operation "." is defined by the generalized Lie group operation of $G - Gd T$.

Proposition 17. The first projection $p: T * M \rightarrow T$ with $w: M \rightarrow T_0$ is a covering morphism of generalized Lie groups. [4]

We construct the category $GLgGdOp(T)$ whose objects are smooth actions (M, w) and a morphism from (M, w) to (M', w') is a smooth map $f: M \rightarrow M'$ such that $w'of = w$ and $f(a.x) = a.f(x)$.

Theorem 18. Let T be a $GLG - Gd$. Then the category $GLgGdCov(T)$ of the coverings of T and the category $GLgGdOp(T)$ of the actions of T on generalized Lie groups are equivalent.

Proof: Firstly, let us define a functor $\Gamma: GLgGdCov(T) \rightarrow GLgGdOp(T)$ in the following way. For any covering morphism $p: \tilde{T} \rightarrow T$ of $GLG - Gd$, we say that $M = \tilde{T}_0$ and $w = p_0: \tilde{T}_0 \rightarrow T_0$. From the Remark 1, we obtain the smooth action $\lambda: T_\alpha \times_w \tilde{T}_0 \rightarrow \tilde{T}_0$, $(a, \tilde{x}) \mapsto a.\tilde{x} = \tilde{\beta}(\tilde{a})$ of $GLG - Gd T$ on $M = \tilde{T}_0$ via the smooth map $w = p_0$. Thus $\Gamma(p)$ is a smooth action of $GLG - Gd T$ on a generalized Lie group.

Secondly, we define a functor $\phi: GLgGdOp(T) \rightarrow GLgGdCov(T)$ in this way. Suppose λ is the action of $GLG - Gd T$ on M via w . From Remark 2, there exists an action $GLG - Gd T * M$ and from Proposition 17 the first projection $p: T * M \rightarrow T$ with $w: M \rightarrow T_0$ is a covering morphism of generalized Lie groups. Thus $\phi(T, w)$ is a covering morphism of $GLG - Gds$.

It is obvious that $\phi\Gamma = 1_{GLgGdCov(T)}$ and $\Gamma\phi = 1_{GLgGdOp(T)}$.

This geometrical categories can be considered on differentiable connections and exponential maps of generalized Lie groups [7, 8].

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