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# Chebyshev cardinal functions: An effective tool for solving nonlinear Volterra and Fredholm integro-differential equations of fractional order

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## Abstract

A computational method for numerical solution of a nonlinear Volterra and Fredholm integro-differential equations of fractional order based on Chebyshev cardinal functions is introduced. The Chebyshev cardinal operational matrix of fractional derivative is derived and used to transform the main equation to a system of algebraic equations. Some examples are included to demonstrate the validity and applicability of the technique.

***Keywords**:* Fractional; Volterra; Fredholm; operational matrix; collocation method of fractional derivative; Caputo derivative; Chebyshev cardinal function

## 1. Introduction

In recent decades, fractional differential equations have gained much attention due to exact description of nonlinear phenomena physical phenomena such as damping laws, electromagnetic, acoustics, viscoelasticity, electroanalytical chemistry neuron modeling, diffusion processing and material sciences [1-2].

In recent years, some attempts have been made to find analytical and numerical solutions for the fractional problems.

These attempts have included finite difference methods [3-5], collocation- shooting methods [6-8], spline and B-spline collocation method [9-10], Adomian decomposition methods [11-12], variational iteration method [13-14], operational matrix methods [15-17] and etc.

In recent years, fractional integro-differential equations have been investigated by many authors [16-22]. Most of the them have utilized linear problems and a small number of their works have considered nonlinear problems.

In this work, we introduce a new operational method to solve nonlinear Volterra and Fredholm integro-differential equations of fractional order. The main characteristic behind the approach using this technique is that it reduces these problems to those of solving a system of algebraic equations, thus greatly simplifying the problem. The Chebyshev cardinal function is used for numerical solution of differential equations, with the goal of obtaining efficient computational solutions. Several papers have appeared in the literature concerned with the application of Chebyshev cardinal functions [23-26].

In this paper, we intend to extend the application of Chebyshev cardinal functions to solve fractional nonlinear Volterra and Fredholm integro-differential equations. Our main aim is to generalize Chebyshev cardinal operational matrix to fractional calculus. It is worthy to mention here that, the method based on using the operational matrix of an interpolate function for solving differential equations is computer oriented.

The rest of the paper is organized as follows: Basic concepts of fractional differential problems are discussed in section 2. Section 3 is devoted to the analysis of the methods and the construction of operational matrix for fractional derivative. Application of the proposed method for fractional problems is given in section 4. The numerical results for effective confirmation of the proposed methods are given in section 5.

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## 2. Concepts of fractional problems

Some basic definitions and properties of the fractional calculus theory used further in this paper are given.

**Definition 2.1.** A real function $f(x), x > 0$, is said to be in the space $C_\mu$, $\mu \in R$, if there exists a real number $p > \mu$ such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0,1)$. Clearly $C_\mu \subset C_\beta$ if $\beta \le \mu$.

**Definition 2.2.** A function $f(x)$, $x > 0$ is said to be in the space $C_\mu^m$, $m \in N \cup \{0\}$, if $f^{(m)} \in C_\mu$.

**Definition 2.3.** The left sided Riemann - Liouville fractional integral operator of order $\alpha \ge 0$ of a function $f \in C_\mu$, $\mu \ge -1$, is defined in [27] as follows:

$$J^{(\alpha)} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad \alpha > 0, \quad x > 0,$$

$$J^{(0)} f(x) = f(x). \tag{1}$$

**Definition 2.4.** Let $f \in C_{-1}^m$, $m \in N \cup \{0\}$. The Caputo fractional derivative of $f(x)$ is defined as in [27]:

$$D^{(\alpha)} f(x) = \begin{cases} J^{(m-\alpha)} f^{(m)}(x), & m-1 < \alpha < m, m \in N, \\ \frac{D^m f(x)}{Dx^m}, & \alpha = m. \end{cases} \tag{2}$$

It can be shown that [27-29]:

1. $J^{(\alpha)} J^{(\nu)} f = J^{(\alpha+\nu)} f$, $\alpha, \nu > 0$, $f \in C\mu$, $\mu > 0$.
2. $J^{(\alpha)} x^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$, $\alpha > 0$, $\gamma > -1$, $x > 0$.
3. $J^{(\alpha)} D^{(\alpha)} f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \dfrac{x^k}{k!}$, $x > 0$, $m-1 < \alpha \le m$.
4. $D^{(\alpha)} J^{(\alpha)} f(x) = f(x)$, $x > 0$, $m-1 < \alpha \le m$,
5. $D^{(\alpha)} C = 0$, $C$ *is constant*
6. $D^{(\alpha)} x^\beta = 0$, $\beta \in N_0$ $\beta < [\alpha]$, $N_0 = \{0,1,\dots\}$
7. $D^{(\alpha)} x^\beta = \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}$,

$$\beta \in N_0 \quad \beta > [\alpha]. \tag{3}$$

The Caputo fractional derivative is considered here because it allows traditional initial and boundary conditions to be included in the formulation of the problem.

## 3. Analysis of the methods

In this section, we describe a brief review of the Chebyshev cardinal functions for solving fractional differential equations.

Chebyshev cardinal functions of order N in $[-1,1]$ are defined as [30]:

$$\phi_j(x) = \frac{T_{N+1}(x)}{T'_{N+1}(x_j)(x-x_j)}, \quad j = 1,2,\dots,N+1, \tag{4}$$

where $T_{N+1}(x)$ is the first kind of Chebyshev function of order $N+1$ in $[-1,1]$ defined by

$$T_{N+1}(x) = \cos\big((N+1)\arccos(x)\big), \tag{5}$$

and $x_j$, $j = 1, 2, \dots, N+1$, are the zeros of $T_{N+1}(x)$ defined by $\cos((2j-1)/(2N+2))$, $j = 1, 2, \dots, N+1$. We apply $t = (x+1)L/2$ to use these functions on $[0, L]$. Now any function $f(t)$ on $[0, L]$ can be approximated as

$$f(t) = \sum_{j=1}^{N+1} f(t_j)\phi_j(t) = F^T \Phi_N(t), \tag{6}$$

where $t_j$, $j = 1, 2, \dots, N+1$, are the shifted points of $x_j$, $j = 1, 2, \dots, N+1$, by transforming $t = \frac{(x+1)L}{2}$ (here we choose $t_j$ so that, $t_1 < t_2 < \cdots < t_{N+1}$), and$F = [f(t_1), f(t_2), \dots, f(t_{N+1})]^T$,

$$\Phi_N(t) = [\phi_1(t), \phi_2(t), \dots, \phi_{N+1}(t)]^T. \tag{7}$$

Note that the functions $\phi_j(t)$ are satisfied in the relation

$$\phi_j(t_i) = \delta_{j,i} = \begin{cases} 1, & j = i, \\ 0, & j \ne i, \end{cases} \quad i = 1, \dots, N+1.$$

So we have

$$\Phi_N(t_i) = e_i, \quad i = 1, \dots, N+1 \tag{8}$$

where $e_i$ is the $i$th column of unit matrix of order $N+1$.

### *3.1. The operational matrix of derivative*

The differentiation of vector $\Phi_N$ in (7) can be expressed as

$$\Phi'_N = \mathbf{D}\Phi_N, \tag{9}$$

where$\mathbf{D}$ is $(N+1)\times(N+1)$operational matrix of derivative for Chebyshev cardinal functions.

It is shown [24] that the matrix $\mathbf{D}$ is the form

$$\mathbf{D} = \begin{bmatrix} \phi'_1(t_1) & \cdots & \phi'_1(t_{N+1}) \\ \vdots & \ddots & \vdots \\ \phi'_{N+1}(t_1) & \cdots & \phi'_{N+1}(t_{N+1}) \end{bmatrix}, \tag{10}$$

where

$$\phi'_j(t_j) = \sum_{\substack{i=1\\ i\neq j}}^{N+1} \frac{1}{t_j - t_i}, \qquad j = 1, \dots, N+1,$$

$$\phi'_j(t_k) = \frac{\beta}{T'_{N+1}(t_j)} \prod_{\substack{l=1\\ l\neq k,j}}^{N+1} (t_k - t_l),$$

$$j, k = 1, \dots, N+1, \quad j \neq k, \tag{11}$$

and $\beta = 2^{2N+1}/L^{N+1}$.

*3.2. The operational matrix of fractional derivative*

The fractional differentiation of vector $\Phi_N(t)$ in (7) can be expressed as

$$D^{(\alpha)}\Phi_N = \mathbf{D}_{\alpha}\Phi_N, \tag{12}$$

where $\mathbf{D}_\alpha$ is $(N+1)\times(N+1)$operational matrix of fractional derivative for Chebyshev cardinal functions. The matrix $\mathbf{D}_\alpha$ can be obtained by the following process. Let

$$D^{(\alpha)}\Phi_N(t) = [\phi_1^{(\alpha)}(t), \phi_2^{(\alpha)}(t), \dots, \phi_{N+1}^{(\alpha)}(t)]^T. \tag{13}$$

Note that

$$\frac{T_{N+1}(t)}{t-t_j} = \beta \times \prod_{\substack{k=1\\ k\neq j}}^{N+1} (t - t_k). \tag{14}$$

Using Eqs. (2), (7) and (14) the function $\phi_j^{(\alpha)}(t)$ can be approximated as

$$\phi_j^{(\alpha)}(t) = \beta \times \frac{1}{T_{N+1}(t_j)} \Big(\prod_{\substack{k=1\\ k\neq j}}^{N+1} (t - t_k)\Big)^{(\alpha)}. \tag{15}$$

Also, we can expand $\prod_{\substack{k=1\\ k\neq j}}^{N+1} (t - t_k)$ as

$$\prod_{\substack{k=1\\ k\neq j}}^{N+1} (t - t_k) = t^N - \left(\sum_{\substack{k_1\neq j\\ 1\leq k_1\leq N+1}} t_{k_1}\right) t^{N-1} + \left(\sum_{\substack{k_1,k_2\neq j\\ 1\leq k_1<k_2\leq N+1}} t_{k_1}t_{k_2}\right) t^{N-2} - \dots + (-1)^N \prod_{\substack{k=1\\ k\neq j}}^{N+1} t_k,$$

$$j = 1,2,\dots,N+1.) \tag{16}$$

**Lemma 3.1.** Let $\phi_n(x)$ be a Chebyshev cardinal function such that $n < \alpha$ then $D^{\alpha}\phi_n(x) = 0$.

**Proof:** Using Eqs. (3) in Eq. (16) the lemma can be proved.

For $0 < \alpha < 1$ using (16), we get

$$\begin{aligned}\phi_j^{(\alpha)}(t) &= \beta \times \frac{1}{T'_{N+1}(t_j)} \Big(\prod_{\substack{k=1\\ k\neq j}}^{N+1} (t - t_k)\Big)^{(\alpha)} \\ &= \frac{\beta}{T'_{N+1}(t_j)\Gamma(N+1-\alpha)} \\ &\times [N!\, t^{N-\alpha} - (\mathrm{N}-\alpha)(N-1)!\,\Big(\sum_{\substack{k_1\neq j\\ 1\leq k_1\leq N+1}} t_{k_1}\Big)t^{N-1-\alpha} \\ &+ (\mathrm{N}-\alpha)(\mathrm{N}-\alpha-1)(N-2)!\,\Big(\sum_{\substack{k_1,k_2\neq j\\ 1\leq k_1<k_2\leq N+1}} t_{k_1}t_{k_2}\Big)t^{N-2-\alpha} - \cdots \\ &+ (-1)^{(N-1)} \prod_{k=0}^{N-2} (\mathrm{N}-\alpha-k) \\ &\times \Big(\sum_{\substack{k_1,k_2,\dots k_{(N-1)}\neq j\\ 1\leq k_1<k_2<\dots<k_{(N-1)}\leq N+1}} t_{k_1}t_{k_2}\dots t_{k_{(N-1)}}\Big)t^{1-\alpha}]\end{aligned}$$

$$j = 1,2,\dots,N+1. \tag{17}$$

Any function $\phi_j^{(\alpha)}(t)$, using (6) can be approximated as

$$\phi_j^{(\alpha)}(t) = \sum_{k=1}^{N+1} \phi_j^{(\alpha)}(t_k)\phi_k(t). \tag{18}$$

Comparing (12) and (18), we get

$$\mathbf{D}_{\alpha} = \begin{bmatrix} \phi_1^{(\alpha)}(t_1) & \cdots & \phi_1^{(\alpha)}(t_{N+1}) \\ \vdots & \ddots & \vdots \\ \phi_{N+1}^{(\alpha)}(t_1) & \cdots & \phi_{N+1}^{(\alpha)}(t_{N+1}) \end{bmatrix}, \tag{19}$$

where the entries of matrix $\mathbf{D}_{\alpha}$ can be found using Eq. (17).

For example for $N = 2$ and $L = 1$, we have

$$\mathbf{D}_{\alpha} = \begin{pmatrix} \frac{32[2!\,t_1^{2-\alpha} - (t_2+t_3)t_1^{1-\alpha}]}{\Gamma(3-\alpha)T'_3(t_1)} & \frac{32[2!\,t_2^{2-\alpha} - (t_2+t_3)t_2^{1-\alpha}]}{\Gamma(3-\alpha)T'_3(t_1)} & \frac{32[2!\,t_3^{2-\alpha} - (t_2+t_3)t_3^{1-\alpha}]}{\Gamma(3-\alpha)T'_3(t_1)} \\ \frac{32[2!\,t_1^{2-\alpha} - (t_1+t_3)t_1^{1-\alpha}]}{\Gamma(3-\alpha)T'_3(t_2)} & \frac{32[2!\,t_2^{2-\alpha} - (t_1+t_3)t_2^{1-\alpha}]}{\Gamma(3-\alpha)T'_3(t_2)} & \frac{32[2!\,t_3^{2-\alpha} - (t_2+t_3)t_3^{1-\alpha}]}{\Gamma(3-\alpha)T'_3(t_2)} \\ \frac{32[2!\,t_1^{2-\alpha} - (t_1+t_2)t_1^{1-\alpha}]}{\Gamma(3-\alpha)T'_3(t_3)} & \frac{32[2!\,t_2^{2-\alpha} - (t_1+t_2)t_2^{1-\alpha}]}{\Gamma(3-\alpha)T'_3(t_3)} & \frac{32[2!\,t_3^{2-\alpha} - (t_1+t_2)t_3^{1-\alpha}]}{\Gamma(3-\alpha)T'_3(t_3)} \end{pmatrix},$$

and so for $\alpha = \frac{1}{2}$, we have

$$\mathbf{D}_{\frac{1}{2}} = \begin{pmatrix} 1.967 & 0.212 & -0.372 \\ -1.418 & 1.418 & 1.418 \\ -0.549 & -1.630 & -1.046 \end{pmatrix}. \tag{20}$$

**Remark 3.2.** *If* $\alpha = n,\quad n \in N,$ then $\quad \mathbf{D}_{\alpha} = \mathbf{D}^{n}$.

*3.3. The operational matrix of integral*

The integral and fractional integral of vector $\Phi_N$ in (7) can be expressed as

$$\int_0^x \Phi_N = \mathbf{I}_{\Phi}\Phi_N, \tag{21}$$

where $\mathbf{I}_{\Phi}$ and $(N+1)\times(N+1)$ operational matrix of integral for Chebyshev cardinal functions. The matrix $I_{\Phi}$ can be obtained by the following process. Let

$$\int_0^x \Phi_N(t)dt = [\int_0^x \phi_1(t)dt, \ldots, \int_0^x \phi_{N+1}(t)dt]^T. \tag{22}$$

Using Eq. (16), any function $\psi_j(x) = \int_0^x \phi_j(t)dt$ can be approximated as

$$\psi_j(x) = \sum_{k=1}^{N+1} \psi_j(t_k)\phi_k(x). \tag{23}$$

Comparing Eqs. (21) and (23), we get

$$\mathbf{I}_{\phi} = \begin{bmatrix} \psi_1(t_1) & \cdots & \psi_1(t_{N+1}) \\ \vdots & \ddots & \vdots \\ \psi_{N+1}(t_1) & \cdots & \psi_{N+1}(t_{N+1}) \end{bmatrix}, \tag{24}$$

where the entries of the matrices $\mathbf{I}_{\phi}$ can be found using Eq. (16) as follows

$$\psi_j(t) = \int_0^x \phi_j(t)dt = \frac{\beta}{T'_{N+1}(t_j)}\Big[\frac{t^N}{N+1} - \Big(\sum_{\substack{k_1\neq j\\ 1\le k_1\le N+1}} t_{k_1}\Big)\frac{t^N}{N} + \Big(\sum_{\substack{k_1,k_2\neq j\\ 1\le k_1<k_2\le N+1}} t_{k_1}t_{k_2}\Big)\frac{t^{N-1}}{N-1} - \cdots$$
$$+ \left(\sum_{\substack{k_1,k_2\neq j\\ 1\le k_1<k_2\le N+1}} t_{k_1}t_{k_2}\right)\frac{t^{N-1}}{N-1} - \cdots + (-1)^N \prod_{\substack{k=1\\ k\neq j}}^{N+1} t_k t\Big],$$

$$j = 1,2,\ldots,N+1. \tag{25}$$

## 4. Application of the operational matrix of fractional derivative

In this section, we apply the operational matrix of fractional derivative to solve nonlinear Volterra and Fredholm integro-differential equations of fractional order.

*4.1. Nonlinear Volterra integro-differential equation of fractional order*

Consider the nonlinear Volterra integro-differential equations of the second kind of fractional order [15]

$$D^{(\alpha)}y(x) - \lambda\int_0^x K(x,s)F(y(s))ds = g(x), 0 \le x < 1, \tag{26}$$

with supplementary conditions

$$y^{(i)}(0) = d_i,\quad i = 0\ldots n,\quad n < [\alpha], \tag{27}$$

where $g \in L^2([0,1))$, $K \in L^2([0,1)^2)$ are known functions, $y(x)$ is unknown function, $D^{(\alpha)}$ is the Caputo fractional differentiation operator of order $\alpha$ and $F(y(s))$ is a polynomial of $y(x)$ with constant coefficients. Moreover, these equations are encountered in combined conduction, convection and radiation problems [27,31,32]. Local and global existence and uniqueness solution of the integro-differential equation given by (26)-(27) is given in [33].

To solve problem (26)-(27), we approximate $y(x)$, $g(x)$ and integral formula in (26) by cardinal Chebyshev functions on [0,1] and define operational matrices as follows

$$D^{(\alpha)}y(x) \approx \Phi_N(x)^T D^{(\alpha)T}C = C^T\,\mathbf{D}_{\alpha}\Phi_N(x),$$

$$g(x) \approx \sum_{i=1}^{N+1} g_i \phi_i(x) = G^T \Phi_N(x),$$
$$l(x,s,y(s)) = K(x,s)F(y(s))$$
$$\approx \sum_{i=1}^{N+1} \sum_{j=1}^{N+1} l\left(t_i, t_j, F\left(y(t_j)\right)\right) \phi_i(x)\phi_j(s)$$
$$= \Phi_N(x)^T \ \mathbf{L}\, \Phi_N(s),$$
$$\int_0^x K(x,s)F(y(s))ds \approx \Phi_N(x)^T \mathbf{L} \int_0^x \Phi_N(s)ds$$
$$= \Phi_N(x)^T \ \mathbf{L}\, \mathbf{I}_\phi \Phi_N(x), \tag{28}$$

where $G = [g_1, \dots, g_{N+1}]^T$,
$g_i = g(t_i), \quad i = 1, \dots, N+1,$
$$l_{i,j} = l\left(t_i, t_j, F\left(C^T \Phi(t_j)\right)\right),$$

$i,j = 1, \dots, N+1$ and $C = [c_1, \dots, c_{N+1}]^T$ is an unknown vector. Employing (28) in (26), we have

$$R_{N+1}(x) = (C^T \ \mathbf{D}_\alpha^{\,T} - \lambda \Phi_N(x)^T \mathbf{L}\mathbf{I}_\phi - G^T)\Phi_N(x) = 0. \tag{29}$$

Collocating Eq.(29) in the points $t_i$, $i = n+2, \dots, N+1$, we get

$$R_{N+1}(t_i) = (C^T \ \mathbf{D}_\alpha^{\,T} - \lambda e_i^T \mathbf{L}\mathbf{I}_\phi - G^T)e_i = 0, \tag{30}$$

where $e_i$ is the $i$th column of unit matrix of order $N+1$. Substituting Eqs. (9) and (28) in Eq. (27), we get

$y(0) \approx C^T \Phi_N(0) = d_0,$
$y'(0) \approx C^T \mathbf{D} \Phi_N(0) = d_1,$
$\vdots$

$$y^{(n)}(0) \approx C^T \mathbf{D}^n \Phi_N(0) = d_n. \tag{31}$$

These equations together with Eq. (30) generate $N+1$ nonlinear equations which can be solved by several methods such as Newton iterative method. Consequently $y(x)$ given in Eq. (26) can be calculated.

*4.2. Nonlinear Fredholm integro-differential equation of fractional order*

Consider the Nonlinear Fredholm integro-differential equation of fractional order

$$D^{(\alpha)} y(x) - \lambda \int_0^1 K(x,s)F(y(s))ds = g(x), \ 0 \le x < 1, \tag{32}$$

with supplementary conditions

$$y^{(i)}(0) = d_i, \quad i = 0 \dots n, \quad n < [\alpha], \tag{33}$$

where $g \in L^2([0,1))$, $K \in L^2([0,1)^2)$ are known functions, $y(x)$ is unknown function, $D^{(\alpha)}$ is the Caputo fractional differentiation operator of order $\alpha$ and $F(y(s))$ is a polynomial of $y(x)$ with constant coefficients.

To solve problem (32) and (33), we approximate $y(x)$ and $g(x)$ by cardinal Chebyshev functions on $[0,1]$ and define operational matrices as follows

$$D^{(\alpha)} y(x) \approx \Phi_N(x)^T D^{(\alpha)T} C = \Phi_N(x)^T \ \mathbf{D}_\alpha^{\,T} C,$$
$$g(x) \approx \sum_{i=1}^{N+1} g_i \phi_i(x) = G^T \Phi_N(x),$$
$$l(x,s,y(s)) = K(x,s)F(y(s)) \approx$$
$$\sum_{i=1}^{N+1} \sum_{j=1}^{N+1} l\left(t_i, t_j, F\left(y(t_j)\right)\right) \phi_i(x)\phi_j(s)$$
$$= \Phi_N(x)^T \ \mathbf{L}\, \Phi_N(s),$$
$$\int_0^1 K(x,s)F(y(s))ds \approx$$
$$\Phi_N(x)^T \ \mathbf{L} \int_0^1 \Phi_N(s)ds = \Phi_N(x)^T \ \mathbf{L}\, \mathbf{H}\Phi_N(x), \tag{34}$$

where $G = [g_1, \dots, g_{N+1}]^T$, $g_i = g(t_i)$,
$i = 1, \dots, N+1$, $l_{i,j} = l\left(t_i, t_j, F\left(C^T \Phi(t_j)\right)\right)$,
$i,j = 1, \dots, N+1$,
$\mathbf{H}$ is$(N+1) \times (N+1)$ operational matrix with $\int_0^1 \phi_i(t)dt = h_{i,j}$, $i,j = 1, \dots, N+1$ and $C = [c_1, \dots, c_{N+1}]^T$ is an unknown vector. Employing (34) in (32), we have

$$R_{N+1}(x) = (C^T \quad D_\alpha^{\,T} - \lambda \Phi_N(x)^T \mathbf{L}\mathbf{H} - G^T)\Phi_N(x) = 0. \tag{35}$$

Collocating Eq. (35) the points $t_i$, $i = n+2, \dots, N+1$, we get

$$R_{N+1}(t_i) = (C^T \ \mathbf{D}_\alpha^{\,T} - \lambda e_i^T \mathbf{L}\mathbf{H} - G^T)e_i = 0. \tag{36}$$

These equations together with Eq. (33) generate $N+1$ nonlinear equations which can be solved. Consequently $y(x)$ given in Eq. (32) can be calculated.

## 5. Numerical examples

In this section, we give the computational results of numerical experiments with methods based on the preceding sections, to support our theoretical discussion. The obtained results by the proposed method compared with the results of [16] in examples 1, 3 and [17] in examples 2, 4. Note that the error of [17] and [16] is shown with $E_j$, where $j$ is size of operational matrix.

**Example 1.** Consider the following fractional nonlinear integro-differential equation [16]

$$D^{\frac{1}{2}} y(x) - \int_0^1 xt\big(y(t)\big)^4 dt = g(x),$$
$$0 \le x < 1, \tag{37}$$

subject to

$$y(0) = 0,$$

where $g(x) = \frac{8/3\, x^{3/2} - 2\sqrt{x}}{\sqrt{\pi}} - \frac{1}{1260}\, x$ with exact solution $y(x) = x^2 - x$.

Figure 1 shows the plot of error with $N = 3, 5, 7, 9$ using the proposed method. This Fig. illustrates that by increasing $N$, the error of results decrease rapidly. In Table 1 the results of the method are compared with the results of [16], which highlights more accuracy of the proposed method. The size of operational matrix in our method is $N + 1$. In Table 1 it is illustrated that the proposed method gives high accuracy with less computational cost compared with [16].

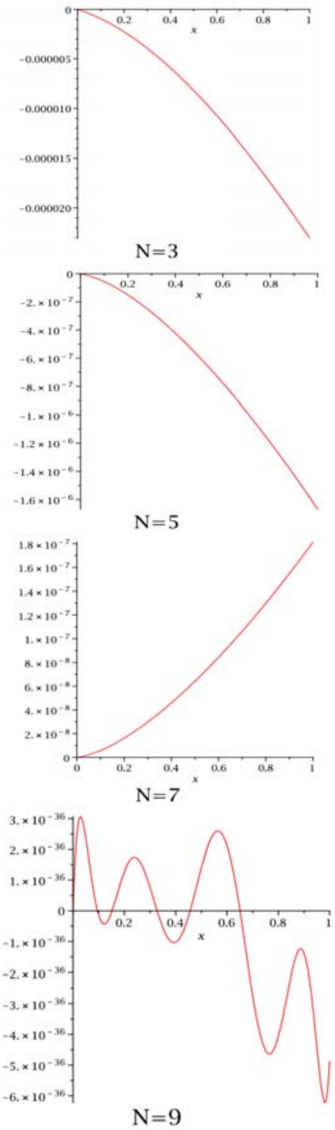



**Fig. 1.** Plot of error for $y(x)$ with $N = 3, 5, 7, 9$ for example 1

**Example 2.** Consider the following equation:

$$D^{\alpha} y(x) + \int_0^x \left(y(t)\right)^2 dt = \sinh(x) + \frac{1}{2}\cosh(x)\sinh(x) - \frac{x}{2},$$
$$0 \le x < 1, \qquad 1 < \alpha \le 2,$$

$$y(0) = 0, \qquad y'(0) = 1. \tag{38}$$

The exact solution of this problem when $\alpha = 2$ is $y(x) = \sinh(x)$. Figure 2 shows the results for $N = 3$ and various $1 < \alpha \le 2$. The comparisions show that as $\alpha \to 2$, the approximate solutions tend to exact solution. The error in the case $\alpha = 2$, for different values of $N$, is shown in Table 1 and Fig. 3. These results show good agreement with the results of references [16, 17, 34].

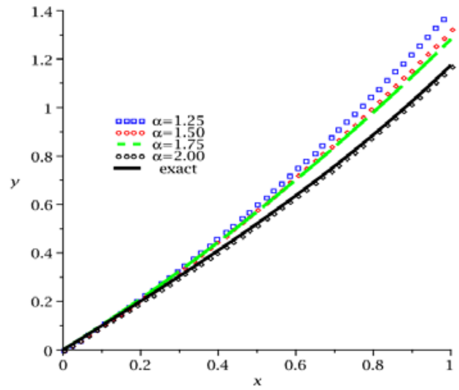
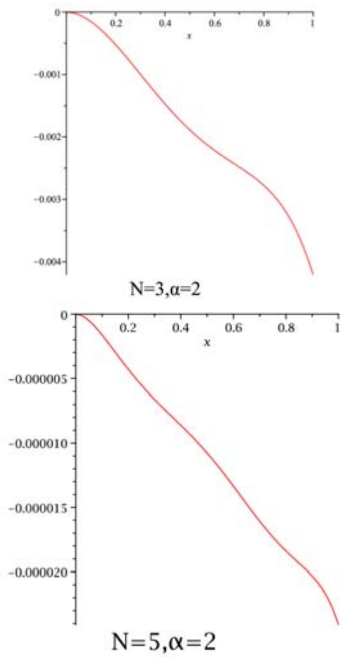



**Fig. 2.** Plot of the approximate solution of example 2 for some $\alpha$ with $N = 3$



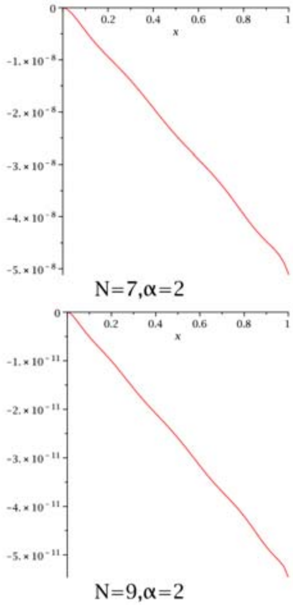



**Fig. 3.** Plot of error for $y(x)$ with $N = 3, 5, 7, 9$ for example 2

**Example 3.** Consider the following nonlinear Fredholm integro-differential equation, of order $\alpha = \frac{5}{3}$: [16]

$$D^{\frac{5}{3}}y(x) - \int_0^1 (x+t)^2 \big(y(t)\big)^3 dt = g(x), \ 0 \le x < 1, \qquad (39)$$

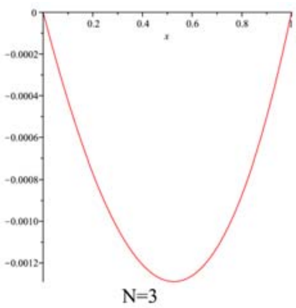
subject to

$$y(0) = 0, \ \ y'(0) = 0$$

where $g(x) = 3\,\frac{\sqrt[3]{x}\sqrt{3}\,\Gamma(2/3)}{\pi} - \frac{x^2}{7} - \frac{x}{4} - \frac{1}{9}$ with exact solution $y(x) = x^2$.

Also, the size of operational matrix in our method, $N+1$, is less than the size of operational matrix in [16], but the accuracy of this approach is higher than the results in [16].

Figure 4 shows the plot of error with $N = 3, 5, 7, 9$ using the proposed method. Similarly example 1, by increasing $N$, the error of results decreases in this Fig. The results of the method compared with the results of [16] in Table 1 that highlights the proposed method more effectively.



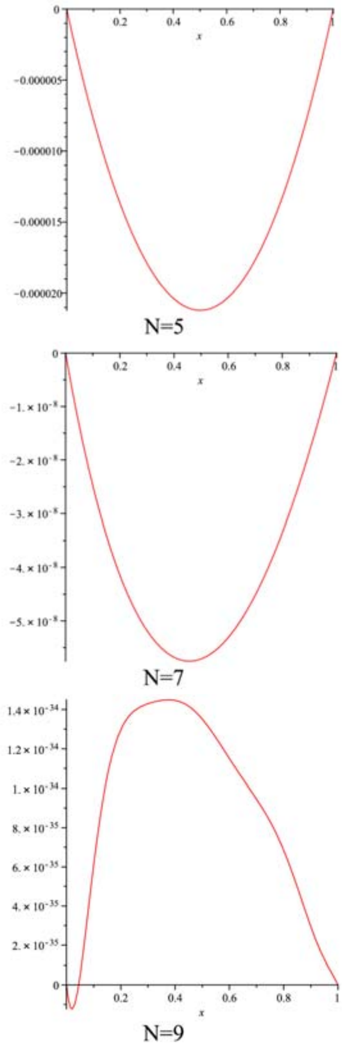




**Fig. 4.** Plot of error for $y(x)$ with $N = 3,5,7,9$ for example 3.

**Table 1.** Approximate of absolute error for $N = 3, 5, 7, 9$

| Examples | $\|\|E_4\|\|_2$ | $\|\|E_6\|\|_2$ | $\|\|E_8\|\|_2$ | $\|\|E_{10}\|\|_2$ | $\|\|E_{12}\|\|_2$ |
|---|---|---|---|---|---|
| | (N=3) | (N=5) | (N=7) | (N=9) | [16,17] |
| 1 | 9.8× e − 6 | 7.0× e − 7 | 7.7× e − 8 | 5.8× e − 36 | 7.7× e − 4 |
| 2 | 1.8× e − 3 | 1.1× e − 5 | 2.4× e − 8 | 2.6× e − 11 | 1.3× e − 6 |
| 3 | 6.4× e − 4 | 1.0× e − 5 | 2.9× e − 8 | 6.7× e − 35 | 3.5× e − 3 |
| 4 | 1.3× e − 2 | 1.9× e − 4 | 7.5× e − 7 | 1.2× e − 9 | 1.6× e − 6 |

**Example 4.** In the following, we consider the fourth order equation:

$$D^{\alpha}y(x) - \int_0^x e^t \big(y(t)\big)^2 dt = 1,$$
$$0 \le x < 1, \qquad 3 < \alpha \le 4,$$
$$y(0) = y'(0) = y''(0) = y'''(0) = 1. \qquad (40)$$

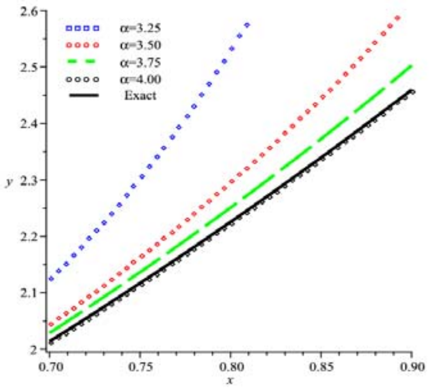
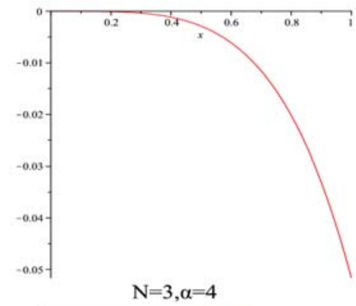
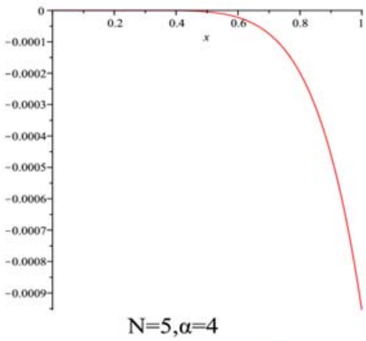
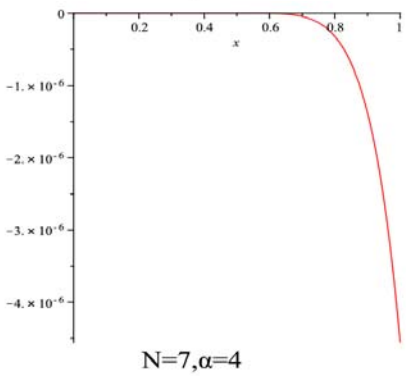



**Fig. 5.** Plot of The approximate solution of Example 4 for some $\alpha$ with $N = 7$.



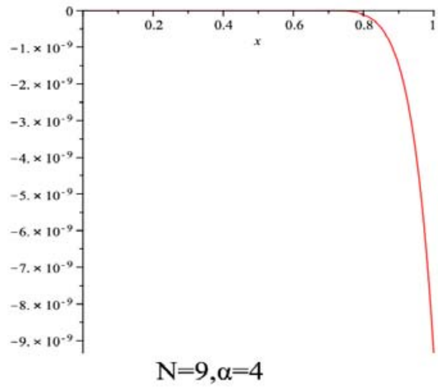




**Fig. 6.** Plot of error for $y(x)$ with $N = 3, 5, 7, 9$ for example 4

The exact solution of this problem when $\alpha = 4$ is $y(x) = e^x$. The numerical results for some $\alpha$ between 3 and 4 are presented in Table 2. This Table shows that the obtained results by the proposed method are similar to Ref. [35, 17]. In Fig. 5, the comparisons show that as $\alpha \to 4$, the approximate solutions tend to exact solution. Figure 6 shows the approximate solution and the plot of error with $N = 3, 5, 7, 9$ when $\alpha = 4$ using proposed method.

**Table 2.** Numerical results for example 4 in $N = 4$ with agreement whth [16, 35]

| $x_i$ | $\alpha =$ 3.25 [35] | $\alpha =$ 3.25[16] | $\alpha =$ 3.25 | $\alpha =$ 3.75[35] | $\alpha =$ 3.75[16] | $\alpha =$ 3.75 |
|---|---|---|---|---|---|---|
| 0 | 1.0 | 1.000004 | 1.000000 | 1.0 | 1.000000 | 1.000000 |
| 0.1 | 1.106551 | 1.105258 | 1.105166 | 1.106151 | 1.105181 | 1.105176 |
| 0.2 | 1.223932 | 1.221892 | 1.221333 | 1.223227 | 1.221452 | 1.221489 |
| 0.3 | 1.353200 | 1.352313 | 1.349500 | 1.352308 | 1.350272 | 1.350292 |
| 0.4 | 1.495601 | 1.496762 | 1.490666 | 1.494636 | 1.492543 | 1.493172 |
| 0.5 | 1.652553 | 1.663409 | 1.645833 | 1.651615 | 1.652178 | 1.651950 |
| 0.6 | 1.825655 | 1.843799 | 1.816000 | 1.824824 | 1.826696 | 1.828683 |
| 0.7 | 2.016687 | 2.044381 | 2.002166 | 2.016024 | 2.109409 | 2.025663 |
| 0.8 | 2.227634 | 2.277591 | 2.205333 | 2.271769 | 2.237195 | 2.245416 |
| 0.9 | 2.460691 | 2.526496 | 2.426500 | 2.460475 | 2.472652 | 2.490704 |

## 6. Conclusion

In this paper we presented a numerical approach for solving the fractional Volterra and Fredholm integro-differential equations. The cardinal Chebyshev functions were employed. The obtained results showed that this approach can solve the problem effectively.

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