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Solution of time fractional diffusion equations using a semi-discrete scheme and collocation method based on Chebyshev polynomials

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Abstract

In this paper, a new numerical method for solving time-fractional diffusion equations is introduced. For this purpose, finite difference scheme for discretization in time and Chebyshev collocation method is applied. Also, to simplify application of the method, the matrix form of the suggested method is obtained. Illustrative examples show that the proposed method is very efficient and accurate.

Keywords*:* Time fractional diffusion equation; finite difference; collocation; chebyshev polynomials

1. Introduction

Fractional partial differential equations (FPDEs) have comprehensive application in the real world. For example, see [1, 2, 3]. For this reason FPDEs have attracted the interest of many researchers.

Most FPDEs do not have an exact analytic solution, thus numerical scheme must be used. The principals of these methods are Chebyshev spectral approximation [4], Walsh function method [5], homotopy perturbation method [6, 7], Adomian decomposition method [8, 9], variational iteration method [7] and generalized differential transform method [10].

One kind of FPDEs is time-fractional partial differential equations (TFPDEs). The analytical solutions of the TFPDEs are studied using Green's functions or Fourier-Laplace transforms. For instance, see [11, 12, 13]. Also Jiang, et al. [14] have used high-order finite element methods for time-fractional partial differential equations.

In this paper, we consider a type of TFPDEs that can be obtained from the standard diffusion equation by replacing the first-order time derivative by Caputo fractional derivative of order $0<\alpha<1$.

More precisely, we study time fractional diffusion equations (TFDEs)

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}-\frac{\partial^2 u(x,t)}{\partial x^2}=p(x,t),\quad 0<x<L,\quad 0<t\le T,\qquad (1)$$

subject to the following initial and boundary conditions:

$$u(x,0)=f(x),\quad 0<x<L,\qquad (2)$$

$$u(0,t)=u(L,t)=0,\quad 0\le t\le T,\qquad (3)$$

where $0<\alpha<1$.

The fractional derivative $\dfrac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ in (1) is the Caputo fractional derivative of order α defined by [3]:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\frac{\partial u(x,\tau)}{\partial \tau}d\tau,\ 0<\alpha<1.$$

Lin and Xu [15] have solved TFDEs by using finite difference and spectral approximation. In addition, Murio [16] has proposed implicit finite difference approximation for solving TFDEs. Also, Crank-Nicolson finite difference method is used by Sweilam and et al. [17] to solve TFDEs.

The aim of this paper is to use a semi-discrete scheme and Chebyshev collocation method for solving time-fractional diffusion equations in the form (1)-(3).

2. Description of the method

In this section, finite difference scheme and Chebyshev collocation method is used for solving TFDEs as in (1)-(3).

2.1. Finite difference scheme for discretization in time

We describe a finite difference method to discretize the time-fractional derivative. Let

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$t_k = k\Delta t,\ k = 0,1,...,M$ where $\Delta t = \dfrac{T}{M}$ is the time step.

The time fractional derivative term can then be approximated by the following formulation [15]: for $k = 0,1,..., M-1$

$$\frac{\partial^\alpha u(x,t_{k+1})}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{k}\int_{t_j}^{t_{j+1}}(t_{k+1}-\tau)^{-\alpha}\frac{\partial u(x,\tau)}{\partial \tau}d\tau$$

$$= \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{k}\frac{u(x,t_{j+1})-u(x,t_j)}{\Delta t}\int_{t_j}^{t_{j+1}}\frac{d\tau}{(t_{k+1}-\tau)^\alpha} + \gamma_{\Delta t}^{k+1}, \quad (4)$$

where $\gamma_{\Delta t}^{k+1}$ is the truncation error. It can be seen that the truncation error takes the following form:

$$\gamma_{\Delta t}^{k+1} \le c_u \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{k}\int_{t_j}^{t_{j+1}}(t_{k+1}-\tau)^{-\alpha}\,(t_{j+1}-t_j-2\tau)d\tau + O(\Delta t^2),$$

where c_u is a constant depending only on u.

Also, from [15] it can be seen that

$$\gamma_{\Delta t}^{k+1} < c_u\,\Delta t^{2-\alpha}.$$

Since $t_k = k\Delta t,\ k = 0,1,...,M$ thus

$$(t_{j+1})^{1-\alpha} - (t_j)^{1-\alpha} = ((j+1)\Delta t)^{1-\alpha} - (j\Delta t)^{1-\alpha}$$
$$= (\Delta t)^{1-\alpha}[(j+1)^{1-\alpha} - j^{1-\alpha}],$$

therefore, from (4) we obtain

$$\frac{\partial^\alpha u(x,t_{k+1})}{\partial t^\alpha} \approx \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{k}\frac{u(x,t_{j+1})-u(x,t_j)}{\Delta t}\int_{t_j}^{t_{j+1}}\frac{d\tau}{(t_{k+1}-\tau)^\alpha}$$
$$= \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{k}\frac{u(x,t_{j+1})-u(x,t_j)}{\Delta t}\int_{t_{k-j}}^{t_{k-j+1}}\frac{d\xi}{\xi^\alpha}$$
$$= \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{k}\frac{u(x,t_{k-j+1})-u(x,t_{k-j})}{\Delta t}\int_{t_j}^{t_{j+1}}\frac{d\xi}{\xi^\alpha}$$
$$= \frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{j=0}^{k}\frac{u(x,t_{k-j+1})-u(x,t_{k-j})}{\Delta t}[(j+1)^{1-\alpha} - j^{1-\alpha}].$$

Let $b_j = (j+1)^{1-\alpha} - j^{1-\alpha}, \quad j = 0,1,..., M$, thus we have

$$\frac{\partial^\alpha u(x,t_{k+1})}{\partial t^\alpha} \approx \frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=0}^{k}b_j\,[u(x,t_{k-j+1})-u(x,t_{k-j})]. \quad (5)$$

Now from (1) and (5) for $k = 0,1,..., M-1$ we obtain:

$$\frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=0}^{k}b_{k-j}\,[u(x,t_{j+1})-u(x,t_j)] - \frac{\partial^2 u(x,t_{k+1})}{\partial x^2} = p(x,t_{k+1}). \quad (6)$$

Let $u(x,t_k) = u_k(x)$, and $p_k(x) = p(x,t_k)$, then from (6) for $k = 0,1,..., M-1$ we have

$$\frac{\Delta t^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=0}^{k}[u_{j+1}(x)-u_j(x)]b_{k-j} - \frac{d^2}{dx^2}u_{k+1}(x) = p_{k+1}(x). \quad (7)$$

2.2. Chebyshev collocation method

First, Chebyshev polynomials and some of their properties are briefly reviewed in this section.

Definition 1. The Chebyshev polynomials of the first kind of degree n are defined on the interval $[-1,1]$ as [18]

$$T_n(x) = \cos(n\arccos(x)).$$

$T_0(x) = 1$, $T_1(x) = x$ and they satisfy the recurrence relations:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),\ n = 1,2,\cdots.$$

In order to use these polynomials on the interval $x \in [0, L]$ we define the so called shifted Chebyshev polynomials by introducing the change of variable $z = \dfrac{2}{L}x - 1$. The shifted Chebyshev polynomials are defined as: $T_n^*(x) = T_n(\dfrac{2}{L}x - 1)$.

Now we expand $u_k(x), k = 0,1,...,M-1$ by shifted Chebyshev polynomials:

$$u_k(x) = \sum_{i=0}^{N} r_i^k\ T_i^*(x),\ k = 1,...,M, \quad (8)$$

where $r_0^k, r_1^k,..., r_N^k$ are unknown coefficients.

For convenience, let $\mu = \Gamma(2-\alpha)\Delta t^\alpha$, then by using (7) and (8) for $k = 0,1,..., M-1$ and some simplifications we obtain

$$\sum_{j=0}^{k}\sum_{i=0}^{N} r_i^{j+1}T_i^*(x)b_{k-j} - \mu\sum_{i=0}^{N} r_i^{k+1}T_i^{*''}(x)$$
$$= \sum_{j=0}^{k}\sum_{i=0}^{N} r_i^{j}T_i^*(x)b_{k-j} + \mu\, p_{k+1}(x). \quad (9)$$

In order to find the unknown coefficients, Chebyshev collocation method with collocation points $x_n = \frac{L}{2}\cos(\frac{n\pi}{N}) + \frac{L}{2}$, $n = 0,1,...,N$ is applied.

$$\sum_{j=0}^{k}\sum_{i=0}^{N} r_i^{j+1}\, T_i^*(x_n) b_{k-j} - \mu \sum_{i=0}^{N} r_i^{k+1}\, T_i^{*''}(x_n)$$
$$= \sum_{j=0}^{k}\sum_{i=0}^{N} r_i^{j}\, T_i^*(x_n) b_{k-j} + \mu\, p_{k+1}(x_n),\ k = 0,1,...,M-1,\ n = 1,...,N-1. \quad (10)$$

Also, boundary conditions (3) for $k = 0,1,...,M-1$ are applied to obtain

$$u_{k+1}(x_0) = \sum_{i=0}^{N} r_i^{k+1}\, T_i^*(x_0) = 0, \quad (11)$$

$$u_{k+1}(x_N) = \sum_{i=0}^{N} r_i^{k+1}\, T_i^*(x_N) = 0. \quad (12)$$

Therefore, eqations (10)-(12) generate a set of (N+1) algebraic equations, which can be solved to find unknown coefficients $r_0^k, r_1^k, ..., r_N^k$.

Remark: Clearly $u_0(x)$ can be obtained from the initial condition as follows:
$u_0(x) = u(x, t_0) = f(x).$

3. The matrix form of the proposed method

In order to find the matrix form of the proposed method, first, by using (10) for $n = 1,2,...,N-1$ we obtain the preliminary matrices **T** and **Q**. Finally, by using (11) and (12) the matrix form for this method is achieved.

By separating the k^th term from the first term of the left hand side of equation (10), for $n = 0,1,...,N-1$, we obtain

$$\sum_{i=0}^{N} r_i^{k+1} T_i^*(x_n) b_0 - \mu \sum_{i=0}^{N} r_i^{k+1} T_i^{*''}(x_n) \quad (13)$$
$$= \sum_{j=0}^{k}\sum_{i=0}^{N} r_i^{j}\, T_i^*(x_n) b_{k-j} - \sum_{j=0}^{k-1}\sum_{i=0}^{N} r_i^{j+1} T_i^*(x_n) b_{k-j} + \mu\, p_{k+1}(x_n).$$

For the terms in RHS of the above equation by making the same upper indices, since $b_0 = 0$ we have

$$\sum_{i=0}^{N} r_i^{k+1} (T_i^*(x_n) - \mu T_i^{*''}(x_n))$$
$$= \sum_{j=0}^{k}\sum_{i=0}^{N} r_i^{j}\, T_i^*(x_n)(b_{k-j} - b_{k-j+1}) + \mu\, p_{k+1}(x_n). \quad (14)$$

Therefore, the matrix form can be obtained for $k = 0,1,...,M-1$ and $n = 1,2,...,N-1$, as follows:

$$\mathbf{T}[r]^{k+1} = \sum_{l=0}^{k}(b_{k-l} - b_{k-l+1})\mathbf{Q}[r]^l + \mu[p]^{k+1}, \quad (15)$$

where

$$[r]^k = [r_0^k, r_1^k, ..., r_N^k]^T, \quad (16)$$

$$[p]^{k+1} = [p_{k+1}(x_1), p_{k+1}(x_2), ..., p_{k+1}(x_{N-1})]^T, \quad (17)$$

and the matrix elements of **T** and **Q** are

$$t_{ij} = \begin{cases} T_{j-1}^*(x_i), & 1 \le i \le N-1,\ j = 1,2, \\ T_{j-1}^*(x_i) - \mu\, T_{j-1}^{*''}(x_i), & 1 \le i \le N-1, 3 \le j \le N+1, \end{cases} \quad (18)$$

$$q_{ij} = T_{j-1}^*(x_i),\ 1 \le i \le N-1,\ 1 \le j \le N+1. \quad (19)$$

Finally, from boundary conditions (11) and (12) the matrix form of the suggested scheme can be obtained

$$\mathbf{A}[r]^{k+1} = \sum_{l=0}^{k}(b_{k-l} - b_{k-l+1})\mathbf{B}[r]^l + [d]^{k+1}, \quad (20)$$

where matrix elements of **A** and **B** are

$$a_{ij} = \begin{cases} T_{j-1}^*(x_0), & i = 1, 1 \le j \le N+1, \\ T_{j-1}^*(x_N), & i = N+1, 1 \le j \le N+1, \\ T_{j-1}^*(x_i), & 1 \le i \le N-1, j = 1,2, \\ T_{j-1}^*(x_i) - \mu T_{j-1}^{*''}(x_i), & 1 \le i \le N-1, 3 \le j \le N+1, \end{cases} \quad (21)$$

$$b_{ij} = \begin{cases} 0, & i = 1, N+1, 1 \le j \le N+1, \\ T_{j-1}^*(x_i), & 1 \le i \le N-1, 1 \le j \le N+1, \end{cases} \quad (22)$$

and

$$[d]^{k+1} = [0, \mu\, p_{k+1}(x_1), \mu\, p_{k+1}(x_2), ..., \mu\, p_{k+1}(x_{N-1}), 0]^T. \quad (23)$$

4. Numerical examples

In order to illustrate the performance of the proposed method in solving time-fractional diffusion equations the following examples are considered. Also, for justifying the accuracy and efficiency of the suggested method, the numerical results thus obtained are compared with other methods.

Remark: In all of the examples, the time step is taken as $\Delta t = 0.001$.

Example 4.1. [17] Consider the following time-fractional diffusion equation

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\partial^2 u(x,t)}{\partial x^2} = 0,\ 0 < x < 1,\ 0 < t \le 1,$$

with the initial condition

$$u(x,0) = sin(\pi x),\ 0 < x < 1$$

and the boundary conditions

$$u(0,t) = u(1,t) = 0,\ 0 \le t \le 1.$$

The exact solution of this problem for $\alpha = 1$ is $u(x,t) = e^{-\pi^2 t}\sin(\pi x)$. The proposed method was applied and the following results were obtained.

In Table 1 absolute error function $|u(x,1) - u_{approx}(x,1)|$ for $\alpha = 1$, $N = 6$ in $x = 0, 0.1, 0.2, ..., 0.9, 1$ is reported. Computer plots for different values of α given in Fig. 1 show that as α approaches to 1, the corresponding solutions of time-fractional diffusion equation in $t = 0.001$ approach the solutions of integer order differential equation.

Note that this problem has been solved in [17] by using Crank-Nicolson finite difference method. In Table 2 we illustrate the magnitude of the maximum error at time $t = 1$ between the exact solution and the numerical solution at different values of $\Delta t = k$ and $\Delta x = h$ by this method.

Table 1. Absolute errors for example 4.1 with N=6 for $\alpha = 1$

x	$\|u(x,1) - u_{approx}(x,1)\|$
0	0
0.1	4.23×10^{-6}
0.2	7.48×10^{-6}
0.3	2.27×10^{-5}
0.4	4.13×10^{-5}
0.5	6.39×10^{-5}
0.6	5.12×10^{-5}
0.7	2.93×10^{-5}
0.8	1.80×10^{-5}
0.9	3.44×10^{-6}
1	0

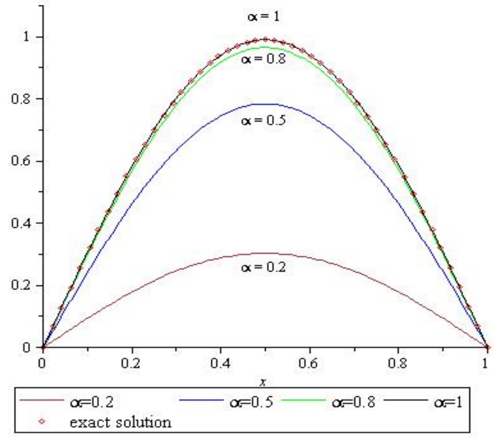

Fig. 1. Approximate solution with N=6 and different values of α for Ex. 4.1

Table 2. Maximum error for the numerical solution using (C-N-FDM) at $t = 1$ for Ex. 4.1.

Δt	Δx	**Maximum error**
0.001	2^{-3}	0.7816×10^{-6}
0.001	2^{-4}	0.2454×10^{-5}
0.002	2^{-5}	0.1969×10^{-6}
0.002	2^{-6}	0.1645×10^{-6}
0.002	2^{-7}	0.1566×10^{-7}

Example 4.2. In this example, we solve numerically the time-fractional diffusion equation (1) with the following initial condition [14]

$$u(x,0) = 0,\ 0 < x < 1,$$

and boundary conditions

$$u(0,t) = u(1,t) = 0,\ 0 \le t \le 1,$$

and the source function

$$p(x,t) = \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha}\sin(2\pi x) + 4\pi^2 t^2 \sin(2\pi x).$$

The exact solution of this problem is $u(x,t) = t^2\sin(2\pi x)$. We solved the problem for $\alpha = 0.5$ by applying the proposed technique described in section 2.

In Table 3, the maximum error between the exact solution and the approximate solution for $N = 3$, $t = 0.1, 0.2, ..., 0.9, 1$ and $0 < x < 1$ is obtained.

Figure 2 shows the exact solution and approximate solution for $u(x,1)$ with $N=3$.

In [14], high-order finite element method has been developed by Jiang and Jingtang to solve this method. Table 4 shows maximum absolute error by using this method for $\alpha=0.5$, $L=20000$ and different values of N.

Table 3. Maximum absolute errors for example 4.2 with N=3 in domain $0<x<1$

x	$max\|u(x,t)-u_{approx}(x,t)$
0.1	5.21×10^{-7}
0.2	8.62×10^{-7}
0.3	8.73×10^{-7}
0.4	5.43×10^{-6}
0.5	7.39×10^{-6}
0.6	9.94×10^{-6}
0.7	2.28×10^{-5}
0.8	3.72×10^{-5}
0.9	4.31×10^{-5}
1	6.50×10^{-5}

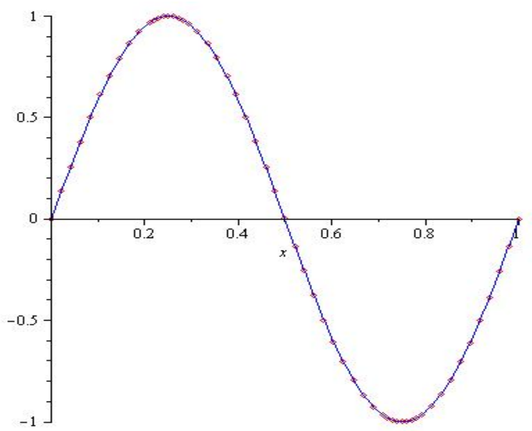

Fig. 2. Approximate solution and exact solution for $u(x,1)$ with $N=3$ for Ex. 4.2

Table 4. Maximum absolute errors for example 4.2 with $\alpha=0.5$ by using the method in [14]

N	10	15	20	25
$\max\|u^n-U^n\|_\infty$	8.01877×10^{-5}	1.5856×10^{-5}	4.9917×10^{-6}	2.0658×10^{-6}
N	**30**	**35**	**40**	
$\max\|u^n-U^n\|_\infty$	9.9666×10^{-7}	5.3572×10^{-7}	3.1174×10^{-7}	

5. Conclusion

In this paper, a semi-discrete scheme for time-fractional diffusion equations by using finite difference and Chebyshev collocation method was studied. Since using matrix form of the method is more convenient for application of collocation method, the matrix form for the suggested method was obtained. The solution of time-fractional diffusion equations by this method is quite satisfactory. The results of numerical examples confirmed the relaibility and efficiency of our method.

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