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Coupled *N*-structures and its application in *BCK/BCI*-algebras

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Abstract

Coupled *N*-structures are introduced, and its application is discussed in *BCK/BCI*- algebras. The notions of a coupled *N*-subalgerba, a coupled *N*-ideal and a coupled *NC*- ideal are introduced, and their relations are investigated. Characterizations of a coupled *N*-ideal and a coupled *NC*-ideal are discussed. Conditions for a coupled *N*-subalgerba to be a coupled *N*-ideal are considered.

Keywords*:* Coupled *N*-structure; coupled *N*-subalgerba; coupled *N*-ideal; coupled *NC*- ideal

1. Introduction

BCK-algebras entered into mathematics in 1966 through the work of Imai and Iséki [1], and have been applied to many branches of mathematics, such as group theory, functional analysis, probability theory and topology. Such algebras generalize Boolean rings as well as Boolean *D*-posets (= *MV* -algebras). Also, Iséki introduced the notion of a BCI-algebra which is a generalization of a BCK-algebra (see [2]). Several properties on BCK/BCI- algebras are investigated in the papers [3-9]. There is a deep relation between BCK/BCI-algebras and posets.

A (crisp) set *A* in a universe *X* can be defined in the form of its characteristic function $\mu_A : X \to \{0,1\}$ yielding the value 1 for elements belonging to the set *A* and the value 0 for elements excluded from the set *A*. So far most of the generalization of the crisp set have been conducted on the unit interval [0, 1] and they are consistent with the asymmetry observation. In other words, the generalization of the crisp set to fuzzy sets relied on spreading positive information that fit the crisp point {1} into the interval [0, 1]. Because no negative meaning of information is suggested, we now feel a need to deal with negative information. To do so, we also feel a need to supply mathematical tool. To attain such object, Jun et al. [10] introduced a new function which is called negative-valued function, and constructed *N*-structures. They discussed *N*-subalgebras and *N*-ideals in BCK/BCI-algebras. Jun et al. [11] applied the *N*-structure to closed ideals in *BCH*-algebras. Also, Jun et al. [12] discuss ideal theory in *BCK/BCI*-algebras based on soft sets and *N*-structures.

In this paper, we introduce the notion of coupled *N*-structures, and discuss its application in *BCK/BCI*-algebras. The notions of a coupled *N*-subalgerba, a coupled *N*-ideal are introduced and a coupled *NC*-ideal, and their relations are investigated. We discuss characterizations of a coupled *N*-ideal and a coupled *NC*-ideal. We provide conditions for a coupled *N*-subalgerba to be a coupled *N*-ideal.

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2. Preliminaries

Let $K(\tau)$ be the class of all algebras of type $\tau = (2, 0)$. By a *BCI-algebra* we mean a system $X := (X, *, 0) \in K(\tau)$ in which the following axioms hold:

$$\text{(a1) } ((x * y) * (x * z)) * (z * y) = 0,$$

$$\text{(a2) } (x * (x * y)) * y = 0,$$

$$\text{(a3) } x * x = 0,$$

$$\text{(a4) } x * y = y * x = 0 \Rightarrow x = y,$$

for all $x, y, z \in X$. We can define a partial ordering $\leq$ by

$$(\forall x, y \in X)\ (x \leq y \Leftrightarrow x * y = 0).$$

In a BCK/BCI-algebra *X*, the following hold:

(b1) $x * 0 = x$,

(b2) $(x * y) * z = (x * z) * y$,

for all $x, y, z \in X$. If a BCI-algebra X satisfies $0 * x = 0$ for all $x \in X$, then we say that X is a *BCK-algebra*. A BCK-algebra X is said to be *commutative* if it satisfies the following equality:

$$(\forall x, y \in X)\ (x \nabla y = y \nabla x) \tag{2.1}$$

where $x \nabla y = x * (x * y)$.

A non-empty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A subset A of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies:

(I1) $0 \in A$,

(I2) $(\forall x, y \in X)\ (x * y \in A,\ y \in A \Rightarrow x \in A)$.

A subset A of a BCK-algebra X is called a *commutative ideal* of X (see [9]) if it satisfies (I1) and

$$(\forall x, y, z \in X)\ ((x * y) * z \in A,\ z \in A \Rightarrow x * (y \nabla x) \in A). \tag{2.2}$$

Note that any commutative ideal in a BCK-algebra is an ideal, but the converse is not valid (see [9]). We refer the reader to the books [13] and [14] for further information regarding BCK/BCI-algebras.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\vee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

$$\wedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

Denote by $F(X, [-1, 0])$ the collection of functions from a set X to $[-1, 0]$. We say that an element of $F(X, [-1, 0])$ is a *negative-valued function* from X to $[-1, 0]$ (briefly, *N-function* on X). By an *N-structure* we mean an ordered pair (X, f) of X and an N-function f on X. We define an order relation "$\ll$" on $[-1, 0]\times[-1, 0]$ as follows:

$$(\forall (r_1, k_1), (r_2, k_2) \in [-1,0] \times [-1,0])\big((r_1, k_1) \ll (r_2, k_2) \Leftrightarrow r_1 \le r_2, k_1 \ge k_2\big).$$

3. Coupled *N*-structures applied to subalgebras and ideals in BCK/BCI-algebras

Definition 3.1. A *coupled N-structure* C in a nonempty set X is an object of the form

$$\mathsf{C} = \{\langle x; f_{\mathsf{C}}, g_{\mathsf{C}}\rangle : x \in X\}$$

where f_{C} and g_{C} are N-functions on X such that $-1 \le f_{\mathsf{C}}(x) + g_{\mathsf{C}}(x) \le 0$ for all $x \in X$.

A coupled N-structure $\mathsf{C} = \{\langle x; f_{\mathsf{C}}, g_{\mathsf{C}}\rangle : x \in X\}$ in X can be identified to an ordered pair $(f_{\mathsf{C}}, g_{\mathsf{C}})$ in $F(X, [-1, 0])\times F(X, [-1, 0])$. For the sake of simplicity, we shall use the notation $\mathsf{C}=(f_{\mathsf{C}}, g_{\mathsf{C}})$ instead of $\mathsf{C} = \{\langle x; f_{\mathsf{C}}, g_{\mathsf{C}}\rangle : x \in X\}$.

For a coupled N-structure $\mathsf{C}=(f_{\mathsf{C}}, g_{\mathsf{C}})$ in X and $t, s \in [-1, 0]$ with $t + s \ge -1$, the set

$$N\{(f_{\mathsf{C}}, g_{\mathsf{C}}); (t, s)\} = \{x \in X \mid f_{\mathsf{C}}(x) \le t,\ g_{\mathsf{C}}(x) \ge s\}$$

is called an *N(t, s)-level set* of $\mathsf{C}=(f_{\mathsf{C}}, g_{\mathsf{C}})$. An $N(t, t)$-level set of $\mathsf{C}=(f_{\mathsf{C}}, g_{\mathsf{C}})$ is called an *N-level set* of $\mathsf{C}=(f_{\mathsf{C}}, g_{\mathsf{C}})$.

Definition 3.2. A coupled N-structure $\mathsf{C}=(f_{\mathsf{C}}, g_{\mathsf{C}})$ in a BCK/BCI-algebra X is called a *coupled N-subalgebra* of X if it satisfies:

$$f_{\mathsf{C}}(x * y) \le \vee\{f_{\mathsf{C}}(x), f_{\mathsf{C}}(y)\} \text{ and } g_{\mathsf{C}}(x * y) \ge \wedge\{g_{\mathsf{C}}(x), g_{\mathsf{C}}(y)\} \tag{3.1}$$

for all $x, y \in X$.

Example 3.3. Let $X = \{0, a, b, c\}$ be a BCK-algebra with the following Cayley Table:

*	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Let $\mathsf{C}=(f_{\mathsf{C}}, g_{\mathsf{C}})$ be a coupled N-structure in X defined by

$$\mathsf{C} = \{\langle 0; -0.6, -0.2\rangle, \langle a; -0.6, -0.2\rangle, \langle b; -0.4, -0.5\rangle, \langle c; -0.6, -0.2\rangle\}.$$

Then $\mathsf{C}=(f_{\mathsf{C}}, g_{\mathsf{C}})$ is a coupled N-subalgebra of X.

Proposition 3.4. *Every coupled N-subalgebra $\mathsf{C}=(f_{\mathsf{C}}, g_{\mathsf{C}})$ of a BCK/BCI-algebra X satisfies the inequalities $f_{\mathsf{C}}(0) \le f_{\mathsf{C}}(x)$ and $g_{\mathsf{C}}(0) \ge g_{\mathsf{C}}(x)$ for all $x \in X$.*

Proof: For any $x, y \in X$, we have

$$f_{\mathsf{C}}(0) = f_{\mathsf{C}}(x * x) \le \vee\{f_{\mathsf{C}}(x), f_{\mathsf{C}}(x)\} = f_{\mathsf{C}}(x),$$
$$g_{\mathsf{C}}(0) = g_{\mathsf{C}}(x * x) \ge \wedge\{g_{\mathsf{C}}(x), g_{\mathsf{C}}(x)\} = g_{\mathsf{C}}(x).$$

This completes the proof.

Using the notion of $N(t, s)$-level sets, we discuss a characterization of a coupled N-subalgebra of a BCK/BCI-algebra X. Although it can be deduced from the so-called transfer principle for fuzzy sets described for BCI/BCK-algebras (see [7, 8]), we provide its detailed proof for the sake of readers.

Theorem 3.5. *A coupled N-structure* C=(f_C, g_C) *in a BCK/BCI-algebra X is a coupled N-subalgebra of X if and only if the nonempty N(t, s)-level set* $N\{(f_C, g_C); (t,s)\}$ *is a subalgebra of X for all t, s*∈[-1, 0] *with* $t+s \geq -1$.

Proof: Assume that C=(f_C, g_C) is a coupled *N*-subalgebra of a BCK/BCI-algebra *X*. Let *t, s*∈[-1, 0] with $t+s \geq -1$ and $x, y \in N\{(f_C, g_C); (t,s)\}$. Then $f_C(x) \leq t, f_C(y) \leq t, g_C(x) \geq s, and\ g_C(y) \geq s$. It follows from (3.1) that

$$f_C(x*y) \leq \vee\{f_C(x), f_C(y)\} \leq t \text{ and } g_C(x*y) \geq \wedge\{g_C(x), g_C(y)\} \geq s$$

so that $x*y \in N\{(f_C, g_C); (t,s)\}$. Hence the nonempty *N*(*t, s*)-level set $N\{(f_C, g_C); (t,s)\}$ is a subalgebra of *X* for all $t, s \in [-1,0]$ with $t+s \geq -1$.

Conversely, suppose that the nonempty *N*(*t, s*)-level set $N\{(f_C, g_C); (t,s)\}$ is a subalgebra of a BCK/BCI-algebra *X* for all $t, s \in [-1,0]$ with $t+s \geq -1$.

Let *x,y*∈*X* be such that $C(x) = (t_x, s_x)$ and $C(y) = (t_y, s_y)$ that is, $f_C(x) = t_x, g_C(x) = s_x, f_C(y) = t_y$ and $g_C(y) = s_y$ with $-1 \leq t_x + s_x$ and $-1 \leq t_y + s_y$. Then $x \in N\{(f_C, g_C); (t_x, s_x)\}$ and $y \in N\{(f_C, g_C); (t_y, s_y)\}$.

We may assume that $(t_x, s_x) \ll (t_y, s_y)$ without loss of generality.
Then

$$N\{(f_C, g_C); (t_x, s_x)\} \subseteq N\{(f_C, g_C); (t_y, s_y)\},$$

and so $x, y \in N\{(f_C, g_C); (t_y, s_y)\}$. Since $N\{(f_C, g_C); (t_y, s_y)\}$ is a subalgebra of *X*, it follows that $x*y \in N\{(f_C, g_C); (t_y, s_y)\}$ so that

$$f_C(x*y) \leq t_y = \vee\{f_C(x), f_C(y)\} \text{ and } g_C(x*y) \geq s_y = \wedge\{g_C(x), g_C(y)\}$$

Therefore C= $(f_C, g_C\}$ in *X* is a coupled *N*-subalgebra of *X*.

Definition 3.6. A coupled *N*-structure C= (f_C, g_C) in a BCK/BCI-algebra *X* is called a *coupled N-ideal* of *X* if it satisfies.

(c1) $f_C(0) \leq f_C(x)$ and $g_C(0) \geq g_C(x)$,
(c2) $f_C(x) \leq \vee\{f_C(x*y), f_C(y)\}$ and $g_C(x) \geq \wedge\{g_C(x*y), g_C(y)\}$

for all $x, y \in X$.

Example 3.7. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley Table:

*	0	*a*	*b*	*c*	*d*
0	0	0	0	0	0
a	*a*	0	*a*	0	0
b	*b*	*b*	0	0	0
c	*c*	*c*	*c*	0	0
d	*d*	*d*	*d*	*a*	0

Let C= (f_C, g_C) be a coupled *N*-structure in *X* defined by

$$C= \{\langle 0; -0.7, -0.2\rangle, \langle a; -0.7, -0.2\rangle, \langle b; -0.7, -0.2\rangle, \langle c; -0.1, -0.6\rangle, \langle d; -0.1, -0.6\rangle\}.$$

Then C= (f_C, g_C) is a coupled *N*-ideal of *X*.

Proposition 3.8. *Every coupled N-ideal of a BCK/BCI-algebra X satisfies the following assertion:*

$$(\forall x, y, z \in X)\ (x*y \leq z \Rightarrow \begin{cases} f_C(x) \leq \vee\{f_C(y), f_C(z)\} \\ g_C(x) \geq \wedge\{g_C(y), g_C(z)\} \end{cases}) \tag{3.2}$$

Proof: Let $x, y, z \in X$ be such that $x*y \leq z$. Then $(x*y)*z = 0$, and so

$$f_C(x) \leq \vee\{f_C(x*y), f_C(y)\} \leq \vee\{\vee\{f_C((x*y)*z), f_C(z)\}, f_C(y)\} = \vee\{\vee\{f_C(0), f_C(z)\}, f_C(y)\} = \vee\{f_C(y), f_C(z)\}$$

and

$$g_C(x) \leq \wedge\{g_C(x*y), g_C(y)\} \leq \wedge\{\wedge\{g_C((x*y)*z), g_C(z)\}, g_C(y) = \wedge\{\wedge\{g_C(0), g_C(z)\}, g_C(y)\} = \wedge\{g_C(y), g_C(z)\}.$$

This completes the proof.

Corollary 3.9. *Every coupled N-ideal of a BCK/BCI-algebra X satisfies the following implication:*

$$(\forall x, y \in X)\big(x \leq y \Rightarrow f_C(x) \leq f_C(y),\ g_C(x) \geq g_C(y)\big). \tag{3.3}$$

Proposition 3.10. *For a coupled N-ideal* C= (f_C, g_C) *of a BCK/BCI-algebra X, the following are equivalent: for any* $x, y \in X$

(1) $(\forall x, y \in X) \begin{pmatrix} f_C(x*y) \leq f_C((x*y)*y) \\ g_C(x*y) \geq g_C((x*y)*y) \end{pmatrix}$.

(2) $(\forall x, y, z \in X) \begin{pmatrix} f_C((x*z)*(y*z))*\leq f_C((x*y)*z) \\ g_C((x*z)*(y*z)) \geq g_C((x*y)*z) \end{pmatrix}$.

Proof: Assume that (1) is valid and let $x, y, z \in X$.

Since

$$\left((x*(y*z))*z\right)*z=((x*z)*(y*z))*z \le (x*y)*z,$$

it follows from (b2), (1) and Corollary 3.9 that

$$f_C((x*z)*(y*z))=f_C\left((x*(y*z))*z\right) \le f_C(((x*(y*z))*z)*z) \le f_C((x*y)*z)$$

and

$$g_C((x*z)*(y*z))=g_C\left((x*(y*z))*z\right) \ge g_C(((x*(y*z))*z)*z) \ge g_C((x*y)*z)$$

Conversely, suppose that (2) holds. If we use z instead of y in (2), then

$$f_C(x*z)=f_C((x*z)*0)=f_C((x*z)*(z*z)) \le f_C((x*z)*z)$$

and

$$g_C(x*z)=g_C((x*z)*0)=g_C((x*z)*(z*z)) \ge g_C((x*z)*z)$$

for all $\forall x,z\in X$ by using (a3) and (b1). This proves (1).

Theorem 3.11. *For a coupled N-structure* C=(f_C,g_C) *in a BCK/BCI-algebra X, the following are equivalent:*
(1) C=(f_C,g_C) *is a coupled N-ideal of X.*
(2) *The nonempty N(t, s)-level set* $N\{(f_C,g_C);(t,s)\}$ *is an ideal of X for all* $t,s\in[-1,0]$ *with* $t+s\ge -1$.

Proof: (1)⇒(2). Obviously, $0\in N\{(f_C,g_C);(t,s)\}$. Let $\forall x,y\in X$ be such that $x*y\in N\{(f_C,g_C);(t,s)\}$ and $y\in N\{(f_C,g_C);(t,s)\}$ for all $t,s\in[-1,0]$ with $t+s\ge-1$. Then $f_C(x*y)\le t, g_C(x*y)\ge s, f_C(y)\le t,$ and $g_C(y)\ge s$. Using (c2), we have $f_C(x)\le \vee\{f_C(x*y),f_C(y)\}\le t$ and $g_C(x)\ge \wedge\{g_C(x*y),g_C(y)\}\ge s$ which imply that $x\in N\{(f_C,g_C);(t,s)\}$. Hence the nonempty $N(t,s)$-level set $N\{(f_C,g_C);(t,s)\}$ is an ideal of X for all $t,s\in[-1,0]$ *with* $t+s\ge-1$.
(2) ⇒ (1). Since $0\in N\{(f_C,g_C);(t,s)\}$, we have the condition (c1). Let $x,y\in X$ be such that $C(x*y)=(t_x,s_x)$ and $C(y)=(t_y,s_y)$, that is, $f_C(x*y)=t_x, g_C(x*y)=s_x, f_C(y)=t_y,$ and $g_C(y)=s_y$.
Then $x*y\in N\{(f_C,g_C);(t_x,s_x)\}$ and $y\in N\{(f_C,g_C);(t_y,s_y)\}$. We may assume that $(t_x,s_x)\ll(t_y,s_y)$ without loss of generality. Then $N\{(f_C,g_C);(t_x,s_x)\}\subseteq N\{(f_C,g_C);(t_y,s_y)\}$, and so $x*y,y\in N\{(f_C,g_C);(t_y,s_y)\}$. Since $N\{(f_C,g_C);(t_y,s_y)\}$ is an ideal of X, it follows that $x\in N\{(f_C,g_C);(t_y,s_y)\}$ so that
$f_C(x)\le t_y=\vee\{f_C(x*y),f_C(y)\}$ and $g_C(x)\ge s_y=\wedge\{g_C(x*y),g_C(y)\}$
Therefore C=(f_C,g_C) in X is a coupled N-ideal of X.

Theorem 3.12. *In a BCK-algebra, every coupled N-ideal is a coupled N-subalgebra.*

Proof: Let C=(f_C,g_C) be a coupled N-ideal of a *BCK*-algebra X. Then the nonempty N-level set $N\{(f_C,g_C);t)\}$ is an ideal of X and so it is a subalgebra of X. It follows from Theorem 3.5 that C=(f_C,g_C) is a coupled N-subalgebra of X.

The following example shows that the converse of Theorem 3.12 is not true.

Example 3.13. Let $X=\{0,1,2,3,4\}$ be a BCK-algebra with the following Cayley Table.

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	1	0
3	3	3	3	0	0
4	4	4	4	3	0

Let C=(f_C,g_C) be a coupled N-structure in X defined by

$$C=\{\langle 0;-0.6,-0.3\rangle,\langle 1;-0.6,-0.3\rangle,\langle 2;-0.4,-0.5\rangle,\langle 3;-0.4,-0.5\rangle,\langle 4;-0.4,-0.5\rangle\}$$

Then C=(f_C,g_C) is a coupled N-subalgebra of X. But it is not a coupled N-ideal of X since

$$f_C(2)=-0.4\nleq -0.6=\vee\{f_C(2*1),f_C(1)\}$$

and/or

$$g_C(2)=-0.5\ngeq -0.3=\wedge\{g_C(2*1),g_C(1)\}.$$

Theorem 3.12. is not true in a *BCI*-algebra as seen in the following example.

Example 3.14. Consider a *BCI*-algebra $X:=Y\times\mathbb{Z}$ where $(Y,*,0)$ is a *BCI*-algebra and $(\mathbb{Z},-,0)$ is the adjoint *BCI*-algebra of the additive group $(\mathbb{Z},+,0)$ of integers (see [13]). Let C=(f_C,g_C) be a coupled *N*-structure in *X* defined by

$$f_C(x) = \begin{cases} t \text{ if } x \in Y \times (\mathrm{N} \cup \{0\}), \\ 0 \text{ otherwise,} \end{cases}$$

$$g_C(x) = \begin{cases} s \text{ if } x \in Y \times (\mathrm{N} \cup \{0\}), \\ 0 \text{ otherwise} \end{cases}$$

for all $x \in X$, where N is the set of all natural numbers and $s, t \in [-1,0]$ with $t + s \geq -1$. One can easily check that $C=(f_C, g_C)$ satisfies the conditions (c1) and (c2). Hence $C=(f_C, g_C)$ is a coupled *N*-ideal of *X*. Take *x* = (0, 0) and *y* = (0, 1). Then $z := x * y = (0,0) * (0,1) = (0,-1)$, and so

$$f_C(x * y) = f_C(z) = 0 \nleq \vee\{f_C(x), f_C(y)\}$$

and/or

$$g_C(x * y) = g_C(z) = 0 \ngeq \wedge\{g_C(x), g_C(y)\}.$$

Therefore $C=(f_C, g_C)$ is not a coupled *N*-subalgebra of *X*.

We now provide a condition for a coupled *N*-subalgebra to be a coupled *N*-ideal.

Theorem 3.15. *Let* $C=(f_C, g_C)$ *be a coupled N-subalgebra of a BCK/BCI-algebra X such that*

$f_C(x) \leq \vee\{f_C(y), f_C(z)\},\ g_C(x) \geq \wedge\{g_C(y), g_C(z)\}$ (3.4)

for all $x, y, z \in X$ *with* $x * y \leq z$. *Then* $C=(f_C, g_C)$ *is a coupled N-ideal of X.*

Proof: Let $C=(f_C, g_C)$ be a coupled *N*-subalgebra of a *BCK/BCI*-algebra *X* satisfying the condition (3.4). Recall from Proposition 3.4 that $f_C(0) \leq f_C(x)$ and $g_C(0) \geq g_C(x)$ for all $x \in X$. Since $x * (x * y) \leq y$ for all $x, y \in X$, it follows from (3.4) that

$$f_C(x) \leq \vee\{f_C(x * y), f_C(y)\},\ g_C(x) \geq \wedge\{g_C(x * y), g_C(y)\}$$

Hence $C=(f_C, g_C)$ is a coupled *N*-ideal of *X*.

For any element *a* of a *BCK/BCI*-algebra *X*, let

$$X_a := \{x \in X \mid f_C(x) \leq f_C(a), g_C(x) \geq g_C(y)\}$$

Obviously, X_a is a non-empty subset of *X*.

Theorem 3.16. *Let a be any element of a BCK/BCI-algebra X. If* $C=(f_C, g_C)$ *is a coupled N-ideal of X, then the set* X_a *is an ideal of X.*

Proof: Obviously, $0 \in X_a$. Let $x, y \in X$ be such that $x * y \in X_a$ and $y \in X_a$. Then $f_C(x * y) \leq f_C(a), g_C(x * y) \geq g_C(a), f_C(y) \leq f_C(a)$ and $g_C(y) \geq g_C(a)$. It follows from (c2) that

$$f_C(x) \leq \vee\{f_C(x * y), f_C(y)\} \leq f_C(a)$$

and

$$g_C(x) \geq \wedge\{g_C(x * y), g_C(y)\} \geq g_C(a)$$

so that $x \in X_a$. Therefore X_a is an ideal of *X*.

Theorem 3.17. *Let a be any element of a BCK/BCI-algebra X and let* $C=(f_C, g_C)$ *be a coupled N-structure in X. Then*

(1) *If* X_a *is an ideal of X, then* $C=(f_C, g_C)$ *satisfies the following assertion:*

$$(\forall x, y, z \in X) \begin{pmatrix} f_C(x) \geq \vee\{f_C(y*z), f_C(z)\} \Rightarrow f_C(x) \geq f_C(y) \\ g_C(x) \leq \wedge\{g_C(y*z), g_C(z)\} \Rightarrow g_C(x) \leq g_C(y) \end{pmatrix}. \quad (3.5)$$

(2) *If* $C=(f_C, g_C)$ *satisfies* (3.5) *and*

$$(\forall x \in X)\ (f_C(0) \leq f_C(x), g_C(0) \geq g_C(x)), \quad (3.6)$$

then X_a *is an ideal of X.*

Proof: (1) Assume that X_a is an ideal of *X* for all $a \in X$. Let $x, y, z \in X$ be such that $f_C(x) \geq \vee\{f_C(y * z), f_C(z)\}$ and $g_C(x) \leq \wedge\{g_C(y * z), g_C(z)\}$. Then $y * z \in X_x$ and $z \in X_x$. Since X_x is an ideal of *X*, it follows that $y \in X_x$ so that $f_C(y) \leq f_C(x)$ and $g_C(y) \geq g_C(x)$.
(2) Suppose that $C=(f_C, g_C)$ satisfies two conditions (3.5) and (3.6). Let $x, y \in X$ be such that $x * y \in X_a$ and $y \in X_a$. Then $f_C(x * y) \leq f_C(a)$, $g_C(x * y) \geq g_C(a)$, $f_C(y) \leq f_C(a)$ and $g_C(y) \geq g_C(a)$. Hence $f_C(a) \geq \vee\{f_C(x * y), f_C(y)\}$ and $g_C(a) \leq \wedge\{g_C(x * y), g_C(y)\}$ which imply from (3.5) that $f_C(a) \geq f_C(x)$ and $g_C(a) \leq g_C(x)$. Thus $x \in X_a$. Obviously, $0 \in X_a$. Therefore X_a is an ideal of *X*.

Definition 3.18. An *N*-structure $C=(f_C, g_C)$ in a *BCI*-algebra *X* is called a *coupled NS-ideal* of *X* if it is both a coupled *N*-subalgebra and a coupled *N*-ideal of *X*.

Example 3.19. Let $X = \{0,1,a,b,c\}$ be a BCI-algebra with the following Cayley Table.

*	0	1	*a*	*b*	*c*
0	0	0	*a*	*b*	*c*
1	1	0	*a*	*b*	*c*
a	*a*	*a*	0	*c*	*b*
b	*b*	*b*	*c*	0	*a*
c	*c*	*c*	*b*	*a*	0

Let $C=(f_C, g_C)$ be a coupled *N*-structure in *X* defined by

$$C = \{\langle 0; -0.8, -0.1\rangle, \langle 1; -0.6, -0.3\rangle, \langle a; -0.5, -0.4\rangle, \langle b; -0.2, -0.7\rangle, \langle c; -0.2, -0.7\rangle\}$$

Then C=(f_C, g_C) is a coupled *Ns*-ideal of *X*.

Theorem 3.20. *Let* C=(f_C, g_C) *be a coupled N-structure in a BCI-algebra X which is given by*

$$f_C(x) := \begin{cases} t_1 & \text{if } x \in X_+, \\ t_2 & \text{otherwise,} \end{cases} \quad g_C(x) := \begin{cases} s_1 & \text{if } x \in X_+, \\ s_2 & \text{otherwise} \end{cases} \quad (3.7)$$

for all $x \in X$, *where* $X_+ = \{x \in X \mid 0 \leq x\}, t_1 < t_2$ *and* $s_1 > s_2$ *in* $[-1,0]$ *with* $-1 \leq t_i + s_i$ *for* $i = 1,2$. *Then* C=(f_C, g_C) *is a coupled NS-ideal of X.*

Proof: Since $0 \in X_+$, we have $f_C(0) = t_1 \leq f_C(x)$ and $g_C(0) = s_1 \geq g_C(x)$ for all $x \in X$. For any $x, y \in X$; if $x \in X_+$ then $f_C(x) = t_1 \leq \vee\{f_C(x * y), f_C(y)\}$ and $g_C(x) = s_1 \geq \wedge\{g_C(x * y), g_C(y)\}$.

Assume that $x \notin X_+$. If $x * y \in X_+$ then $y \notin X_+$, and if $y \in X_+$ then $x * y \notin X_+$. In either case, we get

$f_C(x) = t_2 = \vee\{f_C(x * y), f_C(y)\}$ and $g_C(x) = s_2 = \wedge\{g_C(x * y), g_C(y)\}$.

If any one of *x* and *y* does not belong toX_+, then

$f_C(x) \leq t_2 = \vee\{f_C(x), f_C(y)\}$ and $g_C(x) \geq s_2 = \wedge\{g_C(x), g_C(y)\}$.

If $x, y \in X_+$, then $x * y \in X_+$ and so

$f_C(x) = t_1 = \vee\{f_C(x), f_C(y)\}$ and $g_C(x) = s_1 = \wedge\{g_C(x), g_C(y)\}$.

Therefore C=(f_C, g_C) is a coupled *NS*-ideal of *X*.

For any coupled *N*-structure in a *BCI*-algebra *X*, we consider the next condition.

$$(\forall x \in X)\big(f_C(0 * x) \leq f_C(x), g_C(0 * x) \geq g_C(x)\big). \ (3.8)$$

Proposition 3.21. *Every coupled NS-ideal* C=(f_C, g_C) *in a BCI-algebra X satisfies the condition* (3.8).

Proof: For any $x \in X$, we have

$$f_C(0 * x) \leq \vee\{f_C(0), f_C(x)\} \leq \vee\{f_C(x), f_C(x)\} \leq f_C(x)$$

and

$$g_C(0 * x) \geq \wedge\{g_C(0), g_C(x)\} \geq \wedge\{g_C(x), g_C(x)\} \geq g_C(x)$$

Hence C=(f_C, g_C) satisfies the condition (3.8).

We provide conditions for a coupled *N*-ideal to be a coupled *N*-subalgebra.

Theorem 3.22. *Let* C=(f_C, g_C) *be a coupled N-structure in a BCI-algebra X satisfying the condition* (3.8). *If* C=(f_C, g_C) *is a coupled N-ideal of X, then it is a coupled N- subalgebra of X.*

Proof: Note that $(x * y) * x \leq 0 * y$ for all $x, y \in X$. Using Proposition 3.8 and the condition (3.8), we have

$$f_C(x * y) \leq \vee\{f_C(x), f_C(0 * y)\} \leq \vee\{f_C(x), f_C(y)\}$$

and

$$g_C(x * y) \geq \wedge\{g_C(x), g_C(0 * y)\} \geq \wedge\{g_C(x), g_C(y)\}$$

Therefore C=(f_C, g_C) is a coupled *N*-subalgebra of *X*.

Definition 3.23. Let *X* be a *BCK*-algebra. A coupled *N*-structure C=(f_C, g_C) in *X* is called a *coupled NC-ideal* of *X* if it satisfies the condition (c1) and

$$(\forall x, y, z \in X) \begin{pmatrix} f_C(x*(y\nabla x)) \leq \vee\{f_C((x*y)*z), f_C(z)\} \\ g_C(x*(y\nabla x)) \geq \wedge\{g_C((x*y)*z), g_C(z)\} \end{pmatrix}. \ (3.9)$$

Example 3.24. Consider a *BCK*-algebra *X={0,a,b,c}* which is given in Example 3.3. Let C=(f_C, g_C) be a coupled *N*-structure in *X* defined by

C= $\{\langle 0; -0.6, -0.2\rangle, \langle a; -0.4, -0.4\rangle, \langle b; -0.3, -0.5\rangle, \langle c; -0.3, -0.5\rangle\}$

Routine calculations give that C=(f_C, g_C) is a coupled *NC*-ideal of *X*.

Theorem 3.25. *In a BCK-algebra X, every coupled NC-ideal is a coupled N-ideal.*

Proof: Let C=(f_C, g_C) be a coupled *NC*-ideal of a *BCK*-algebra *X*. Let $\forall x, y, z \in X$. Using (3.9) and (b1), we have

$$f_C(x) = f_C(x * (0\nabla x)) \leq \vee\{f_C((x * 0) * z), f_C(z)\} = \vee\{f_C(x * z), f_C(z)\}$$

and

$$g_C(x) = g_C(x * (0\nabla x)) \leq \wedge\{g_C((x * 0) * z), g_C(z)\} = \wedge\{g_C(x * z), g_C(z)\}$$

Hence C=(f_C, g_C) is a coupled *N*-ideal of *X*.

The following example shows that the converse of Theorem 3.25 is not true.

Example 3.26. Let $X = \{0,1,2,3,4\}$ be a BCK-algebra with the following Cayley Table:

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	3	0

Let C=(f_C, g_C) be a coupled *N*-structure given by

$$C = \{\langle 0; -0.7, -0.25\rangle, \langle 1; -0.6, -0.35\rangle, \langle 2; -0.4, -0.45\rangle, \langle 3; -0.4, -0.45\rangle, \langle 4; -0.4, -0.45\rangle\}$$

Then C=(f_C, g_C) is a coupled *N*-ideal of *X*, but it is not a coupled *NC*-ideal of *X* since

$$f_C(2 * (3\nabla 2)) = f_C(2) = -0.4 > -0.7 = \vee\{f_C((2*3)*0), f_C(0)\}$$

and/or

$$g_C(2 * (3\nabla 2)) = g_C(2) = -0.45 < -0.25 = \wedge\{g_C((2*3)*0), g_C(0)\}$$

Theorem 3.27. *Let X be a BCK-algebra. A coupled N-structure* C=(f_C, g_C) *in X is a coupled NC-ideal of X if and only if* C=(f_C, g_C) *is a coupled N-ideal of X that satisfies:*

$$(\forall x, y, z \in X)\ \begin{pmatrix} f_C(x*y) \geq f_C(x*(y\nabla x)) \\ g_C(x*y) \leq g_C(x*(y\nabla x)) \end{pmatrix}. \quad (3.10)$$

Proof: Assume that C=(f_C, g_C) is a coupled *NC*-ideal of *X*. Then C=(f_C, g_C) is a coupled *N*-ideal of *X* by Theorem 3.25. Taking $z = 0$ in (3.9) and using (c1) and (b1) induces (3.10).

Conversely, let C=(f_C, g_C) be a coupled *N*-ideal of a *BCK*-algebra *X* that satisfies the condition (3.10). Then we have

$$(\forall x, y, z \in X)\ \begin{pmatrix} f_C(x*y) \leq \vee\{f_C((x*y)*z), f_C(z)\} \\ g_C(x*y) \geq \wedge\{g_C((x*y)*z), g_C(z)\} \end{pmatrix}. \quad (3.11)$$

Combining (3.10) and (3.11) yields (3.9). Hence C=(f_C, g_C) is a coupled *NC*-ideal of *X*.

Theorem 3.28. *In a commutative BCK-algebra, every coupled N-ideal is a coupled NC-ideal.*

Proof: Let C=(f_C, g_C) be a coupled *N*-ideal of a commutative *BCK*-algebra *X*. Since *X* is commutative, it follows from (a1) and (b2) that

$$\big((x*(y\nabla x)) * ((x*y)*z)\big) * z = \big((x*(y\nabla x)) * z\big) * \big((x*y)*z\big) \leq (x*(y\nabla x)) * (x*y) = (x\nabla y) * (y\nabla x) = 0$$

so that $\big((x*(y\nabla x)) * ((x*y)*z)\big) * z = 0$, i.e., $\big(x*(y\nabla x)\big) * ((x*y)*z) \leq z$ for all $x, y, z \in X$. Since C=(f_C, g_C) is a coupled *N*-ideal, we have

$$f_C\big(x*(y\nabla x)\big) \leq \vee\{f_C\big((x*y)*z\big), f_C(z)\}$$

and

$$g_C\big(x*(y\nabla x)\big) \geq \wedge\{g_C\big((x*y)*z\big), g_C(z)\}$$

for all $x, y, z \in X$ by Proposition 3.8. Therefore C=(f_C, g_C) is a coupled *NC*-ideal of *X*.

Lemma 3.29. [14] *An ideal A of a BCK-algebra X is commutative if and only if the following implication is valid:*

$$(\forall x, y \in X)(x*y \in A \Rightarrow x*(y\nabla x) \in A).$$

Theorem 3.30. *For a coupled N-structure* C=(f_C, g_C) *in a BCK-algebra X, the following are equivalent:*

(1) C=(f_C, g_C) *is a coupled NC-ideal of X:*
(2) *The nonempty N(t, s)-level set of* C=(f_C, g_C) *is a commutative ideal of X for all t, s*∈[-1, 0] *with* $t + s \geq -1$.

Proof: Assume that C=(f_C, g_C) is a coupled *NC*-ideal of *X*. Let *t, s*∈[-1, 0] be such that $t + s \geq -1$. Then C=(f_C, g_C) is a coupled *N*-ideal of *X* by

Theorem 3.25. and so the nonempty *N*(*t, s*)-level set of C=(f_C, g_C) is an ideal of *X* by Theorem 3.11. Let $x, y \in X$ be such that $x*y \in N\{(f_C, g_C); (t, s)\}$. Then $f_C(x*y) \leq t$ and $g_C(x*y) \geq s$. It follows from (3.10) that
$f_C(x*(y\nabla x)) \leq f_C(x*y) \leq t$ and $g_C(x*(y\nabla x)) \geq g_C(x*y) \geq s$.
Therefore $x*(y\nabla x) \in N\{(f_C, g_C); (t, s)\}$. Using

Lemma 3.29. we conclude that the nonempty *N*(*t, s*)-level set of C=(f_C, g_C) is a commutative ideal of *X* for *all t, s*∈[-1, 0] *with* $t + s \geq -1$.

Conversely, suppose that the nonempty *N*(*t, s*)-level set of C=(f_C, g_C) is a com-mutative ideal of *X* for all *t, s*∈[-1, 0] *with* $t + s \geq -1$. Then it is an ideal of *X*, and so C=(f_C, g_C) is a coupled *N*-ideal of *X* by

Theorem 3.11. Assume that there exist $a, b, c \in X$ such that $f_C(a*b) < f_C(a*(b\nabla a))$ or $g_C(a*b) > g_C(a*(b\nabla a))$. For the case $f_C(a*b) < f_C(a*(b\nabla a))$ and $g_C(a*b) \leq g_C(a*(b\nabla a))$, let $t_0 := \frac{1}{2}\big(f_C(a*b) + f_C\big(a*(b\nabla a)\big)\big)$ and $s_0 := g_C(a*b)$. Then

$a * b \in N\{(f_C, g_C); (t_0, s_0)\}$, but $a * (b\nabla a) \notin N\{(f_C, g_C); (t_0, s_0)\}$. For the case $f_C(a * b) \geq f_C(a * (b\nabla a))$ and $g_C(a * b) > g_C(a * (b\nabla a))$, let $t_0 := f_C(a * b)$ and $s_0 := \frac{1}{2}\left(g_C(a * b) + g_C(a * (b\nabla a))\right)$. Then $a * b \in N\{(f_C, g_C); (t_0, s_0)\}$, but $a * (b\nabla a) \notin N\{(f_C, g_C); (t_0, s_0)\}$. If $f_C(a * b) < f_C(a * (b\nabla a))$ and $g_C(a * b) > g_C(a * (b\nabla a))$, then $a * b \in N\{(f_C, g_C); (t_0, s_0)\}$ but $a * (b\nabla a) \notin N\{(f_C, g_C); (t_0, s_0)\}$ where $t_0 := \frac{1}{2}\left(f_C(a * b) + f_C(a * (b\nabla a))\right)$ and $s_0 := \frac{1}{2}\left(g_C(a * b) + g_C(a * (b\nabla a))\right)$. This is a contradiction, and so (3.10) is valid. Therefore C=(f_C, g_C) is a coupled N_C-ideal of *X* by Theorem 3.27.

Theorem 3.31. *Let a be any element of a BCK-algebra X. If* C=(f_C, g_C) *is a coupled NC-ideal of X, then the set*

$$X_a := \{x \in X \mid f_C(x) \leq f_C(a),\ g_C(x) \geq g_C(a)\}$$

is a commutative ideal of X.

Proof: If C=(f_C, g_C) is a coupled *NC*-ideal of *X*, then it is a coupled *N*-ideal of *X* by Theorem 3.25. Hence *Xa* is an ideal of *X* by Theorem 3.16. Let $x, y \in X$ be such that $x * y \in X_a$; Then $f_C(x * y) \leq f_C(a)$ and $g_C(x * y) \geq g_C(a)$. It follows from (3.10) that $f_C(x * (y\nabla x)) \leq f_C(x * y) \leq f_C(a)$ and $g_C(x * y\nabla x \geq gC(x*y) \geq gC(a)$ so that $x*y\nabla x \in Xa$. Using Lemma 3.29, we know that *Xa* is a commutative ideal of *X*.

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