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Characterizations of semigroups by the properties of their $(\in_\gamma,\in_\gamma \vee q_\delta)$-fuzzy ideals

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Abstract

Generalizing the notions of $(\in,\in\vee q)$-fuzzy left (right) ideal, $(\in,\in\vee q)$-fuzzy quasi-ideal, and $(\in,\in\vee q)$-fuzzy bi-ideal, the notions of $(\in_\gamma,\in_\gamma \vee q_\delta)$-fuzzy left (right) ideal, $(\in_\gamma,\in_\gamma \vee q_\delta)$-fuzzy quasi-ideal and $(\in_\gamma,\in_\gamma \vee q_\delta)$-fuzzy bi-ideal of semigroups are defined. Regular, intra regular and semisimple semigroups are characterized by the properties of these fuzzy ideals.

***Keywords**:* $(\in_\gamma,\in_\gamma \vee q_\delta)$-fuzzy ideal; $(\in_\gamma,\in_\gamma \vee q_\delta)$-fuzzy quasi-ideal; $(\in_\gamma,\in_\gamma \vee q_\delta)$-fuzzy bi-ideal; regular semigroups; intra regular semigroups; semisimple semigroups

1. Introduction

Many researchers used the concept of fuzzy set, introduced by Zadeh [1] in 1965, to generalize some of the notions of algebra. Rosenfeld [2] laid the foundations of fuzzy algebra in 1971. He introduced the notion of fuzzy subgroup (subgroupoid) of a group (groupoid). Kuroki [3, 4] initiated the study of fuzzy semigroups. Bhakat and Das [5, 6] used the "belongs to" relation and "quasi-coincidence with" relation, given in [7, 8], and defined $(\in,\in\vee q)$-fuzzy subgroups which are generalizations of Rosenfeld fuzzy subgroups. Many authors applied this idea to define $(\in,\in\vee q)$-fuzzy substructures of different algebraic structures (see [9-20]). Generalizing the concept of the quasi-coincidence of a fuzzy point with a fuzzy set, Jun [21] defined $(\in,\in\vee q_k)$-fuzzy subalgebras in BCK/ BCI-algebras. In [22] Shabir et al. characterized semigroups by the properties of $(\in,\in\vee q_k)$-fuzzy ideals, $(\in,\in\vee q_k)$-fuzzy quasi-ideals and $(\in,\in\vee q_k)$-fuzzy bi-ideals. Recently, Shabir and Rehman [23], studied $(\in,\in\vee q_k)$-fuzzy ideals of ternary semigroups. In this paper, generalizing the notions of $(\in,\in\vee q)$-fuzzy left (right) ideal, $(\in,\in\vee q)$-fuzzy quasi-ideal, and $(\in,\in\vee q)$-fuzzy bi-ideal, the notions of $(\in_\gamma,\in_\gamma \vee q_\delta)$-fuzzy left (right) ideal, $(\in_\gamma,\in_\gamma \vee q_\delta)$-fuzzy quasi-ideal and $(\in_\gamma,\in_\gamma \vee q_\delta)$-fuzzy bi-ideal of semigroup are defined. Also, regular, intra regular, and semisimple semigroups are characterized by the properties of these fuzzy ideals.

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2. Preliminaries

An algebraic system $(S,.)$ consisting of a non-empty set S together with an associative binary operation "." is called a semigroup. A non-empty subset A of a semigroup S is called a subsemigroup of S if $ab\in A$ for all $a,b\in A$, that is $A^2\subseteq A$. A non-empty subset A of a semigroup S is called a left (right) ideal of S if $sa\in A$ $(as\in A)$ for all $a\in A$ and $s\in S$. A is called a two sided ideal or simply an ideal of S if it is both a left ideal and a right ideal of S. A non-empty subset Q of a semigroup S is called a quasi-ideal of S if $QS\cap SQ\subseteq Q$. A subsemigroup B of a semigroup S is called a bi-ideal of S if $BSB \subseteq B$. A non-empty subset B of a semigroup S is called a generalized bi-ideal of S if $BSB \subseteq B$. A non-empty subset I of a semigroup S is called an interior ideal of S if $SIS \subseteq S$. Every ideal of a semigroup S is an interior ideal of S but the converse is not true. Every left (right) ideal of a semigroup S is a quasi-ideal. Every quasi-ideal is a bi-ideal and every bi-ideal is a generalized bi-ideal of S. But the converse is not true.

A semigroup S is called regular if for each $x \in S$ there exists $a \in S$ such that $x = xax$. A semigroup S is called intra regular if for each $x \in S$ there exist $a, b \in S$ such that $x = ax^2 b$. In general, neither regular semigroup is intra regular nor is intra regular semigroup regular. But in commutative semigroups both the concepts coincide. A semigroup S is called semisimple if every ideal of S is idempotent. It is clear that S is semisimple if and only if $x \in (SxS)(SxS)$ for every $x \in S$, that is there exist $a, b, c \in S$ such that $x = axbxc$. In general, every regular semigroup is semisimple but the converse is not true. However in a commutative semigroup both the concepts coincide.

The following results are well known.

2.1. Theorem The following assertions for a semigroup S are equivalent.
(1) S is regular.
(2) $RL = R \cap L$ for every right ideal R and left ideal L of S.
(3) $Q = QSQ$ for every quasi-ideal Q of S.

2.2. Theorem The following assertions for a semigroup S are equivalent.
(1) S is intra regular.
(2) $R \cap L \subseteq LR$ for every right ideal R and left ideal L of S.

2.3. Theorem The following assertions for a semigroup S are equivalent.
(1) S is both regular and intra regular.
(2) Every quasi-ideal of S is idempotent.
(3) Every bi-ideal of S is idempotent.

A fuzzy subset f of a set X is a function from X into the unit closed interval $[0,1]$, that is $f : X \to [0,1]$. If f and g are fuzzy subsets of X, then $f \le g$ means that $f(x) \le g(x)$ for all $x \in X$. The fuzzy subsets $f \wedge g$ and $f \vee g$ of X are defined as $(f \wedge g)(x) = f(x) \wedge g(x)$ and $(f \vee g)(x) = f(x) \vee g(x)$ for all $x \in X$. If $\{f_i\}_{i \in I}$ is a family of fuzzy subsets of X, then $\underset{i \in I}{\wedge} f_i$ and $\underset{i \in I}{\vee} f_i$ are fuzzy subsets of X defined by $\left(\underset{i \in I}{\wedge} f_i\right)(x) = \inf\{f_i(x)\}_{i \in I}$ and $\left(\underset{i \in I}{\vee} f_i\right)(x) = \sup\{f_i(x)\}_{i \in I}$ for all $x \in X$.

Let f be a fuzzy subset of X and $t \in (0,1]$. Then

$$U(f;t) = \{x \in S \ : \ f(x) \ge t\}$$

is called the level subset of f.

A fuzzy subset f of X of the form

$$f\ (a) \begin{cases} t \ne 0 \text{ if } a = x \\ 0 \quad \text{otherwise} \end{cases}$$

is called a fuzzy point with support x and value t and is denoted by x_t.

A fuzzy point x_t "belongs to" (resp. "quasi-coincident with") a fuzzy set f, written as $x_t \in f$ (resp. $x_t q f$) if $f(x) \ge t$ (resp. $f(x) + t > 1$) (cf. [8]). If $x_t \in f$ or $x_t q f$, then we write $x_t \in \vee q f$. If $x_t \in f$ and $x_t q f$ then we write $x_t \in \wedge q f$. If $f(x) < t$ (resp. $f(x) + t \le 1$), then we say that $x_t \overline{\in} f$ (resp. $x_t \overline{q} f$). Similarly $\overline{\in \vee q}$ (resp. $\overline{\in \wedge q}$) means that $\in \vee q$ (resp. $\in \wedge q$) does not hold.

Let $\gamma, \delta \in [0,1]$ be such that $\gamma < \delta$. For a fuzzy point x_t and a fuzzy subset f of X, we say
(1) $x_t \in_\gamma f$ if $f(x) \ge t > \gamma$.
(2) $x_t q_\delta f$ if $f(x) + t > 2\delta$.
(3) $x_t \in_\gamma \vee q_\delta f$ if $x_t \in_\gamma f$ or $x_t q_\delta f$.
(4) $x_t \in_\gamma \wedge q_\delta f$ if $x_t \in_\gamma f$ and $x_t q_\delta f$.
(5) $x_t \overline{\alpha} f$ if $x_t \alpha f$ does not hold for $\alpha \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$.

Let f and g be fuzzy subsets of a semigroup S. Then their product fg is a fuzzy subset of S

defined by

$$(fg)(x)=\begin{cases}\bigvee_{x=yz}\{f(y)\wedge g(z)\} & \text{if } x \text{ is expressible as } x=yz \text{ for some } y,z\in S\\ 0 & \text{otherwise.}\end{cases}$$

3. (α,β)-fuzzy ideals

Throughout the remaining paper $\gamma,\delta\in[0,1]$, where $\gamma<\delta$. $\alpha,\beta\in\{\in_\gamma, q_\delta, \in_\gamma\vee q_\delta, \in_\gamma\wedge q_\delta\}$ and $\alpha\neq\in_\gamma\wedge q_\delta$.

Let f be a fuzzy subset of a semigroup S such that $f(x)\le\delta$. Let $x\in S$ and $t\in[0,1]$ be such that $x_t\in_\gamma\wedge q_\delta f$. Then $f(x)\ge t>\gamma$ and $f(x)+t>2\delta$. It follows that $2\delta<f(x)+t\le f(x)+f(x)=2f(x)$, that is $f(x)>\delta$. This means that $\{x_t : x_t\in_\gamma\wedge q_\delta f\}=\phi$. Therefore we are not taking $\alpha=\in_\gamma\wedge q_\delta$.

3.1. Definition A fuzzy subset f of a semigroup S is called an (α,β)-fuzzy subsemigroup of S, if it satisfies

(F1) $x_t\alpha f$ and $y_r\alpha f\Rightarrow(xy)_{\min\{t,r\}}\beta f$ for all $x,y\in S$ and $t,r\in(\gamma,1]$.

3.2. Definition A fuzzy subset f of a semigroup S is called an (α,β)-fuzzy left (right) ideal of S, if it satisfies

(F2) $x_t\alpha f \Rightarrow (yx)_t\beta f$ (resp. $(xy)_t\beta f$) for all $x,y\in S$ and $t\in(\gamma,1]$.

A fuzzy subset f of a semigroup S is called an (α,β)-fuzzy ideal of S, if it is both (α,β)-fuzzy left ideal and (α,β)-fuzzy right ideal of S.

3.3. Definition A fuzzy subset f of a semigroup S is called an (α,β)-fuzzy interior ideal of S, if it satisfies

(F3) $x_t\alpha f \Rightarrow (yxz)_t\beta f$ for all $x,y,z\in S$ and $t\in(\gamma,1]$.

3.4. Definition A fuzzy subset f of a semigroup S is called an (α,β)-fuzzy generalized bi-ideal of S, if it satisfies

(F4) $x_t\alpha f$ and $y_r\alpha f\Rightarrow(xzy)_{\min\{t,r\}}\beta f$ for all $x,y,z\in S$ and $t,r\in(\gamma,1]$.

3.5. Definition A fuzzy subset f of a semigroup S is called an (α,β)-fuzzy bi-ideal of S, if it satisfies conditions (F1) and (F4).

3.6. Theorem Let $2\delta=1+\gamma$ and f be an (α,β)-fuzzy subsemigroup of S. Then $f_\gamma=\{x\in S : f(x)>\gamma\}$ is a subsemigroup of S.

Proof: Let $x,y\in f_\gamma$. Then $f(x)>\gamma$ and $f(y)>\gamma$. Suppose that $f(xy)\le\gamma$. If $\alpha\in\{\in_\gamma,\in_\gamma\vee q_\delta\}$, then $x_{f(x)}\alpha f$ and $y_{f(x)}\alpha f$ but $(xy)_{\min\{f(x),f(y)\}}\overline{\beta}f$ for every $\beta\in\{\in_\gamma, q_\delta, \in_\gamma\vee q_\delta, \in_\gamma\wedge q_\delta\}$ (because $f(xy)\le\gamma<\min\{f(x),f(y)\}$, so $(xy)_{\min\{f(x),f(y)\}}\overline{\in}f$ and $f(xy)+\min\{f(x),f(y)\}\le\gamma+\min\{f(x),f(y)\}\le\gamma+1=2\delta$, so $(xy)_{\min\{f(x),f(y)\}}\overline{q_\delta}f$), a contradiction. Hence $f(xy)>\gamma$, that is $xy\in f_\gamma$. If $\alpha=q_\delta$ then $x_1q_\delta f$ and $y_1q_\delta f$ (because $f(x)+1>1+\gamma=2\delta$ and $f(y)+1>1+\gamma=2\delta$). But $(xy)_1\overline{\beta}f$ for every $\beta\in\{\in_\gamma, q_\delta, \in_\gamma\vee q_\delta, \in_\gamma\wedge q_\delta\}$ (because $f(xy)\le\gamma$, so $(xy)_1\overline{\in}f$ and $f(xy)+1\le\gamma+1=2\delta$, so $(xy)_1\overline{q_\delta}f$), a contradiction. Hence $f(xy)>\gamma$, that is $xy\in f_\gamma$. This shows that f_γ is a subsemigroup of S.

3.7. Theorem Let $2\delta=1+\gamma$ and f be an (α,β)-fuzzy left (right) ideal of S. Then $f_\gamma=\{x\in S : f(x)>\gamma\}$ is a left (right) ideal of S.

Proof: The proof is similar to the proof of Theorem 3.6.

3.8. Theorem (1) Let $2\delta=1+\gamma$ and f be an (α,β)-fuzzy generalized bi-ideal of S. Then f_γ is a generalized bi-ideal of S.

(2) Let $2\delta=1+\gamma$ and f be an (α,β)-fuzzy bi-ideal of S. Then f_γ is a bi-ideal of S.

(3) Let $2\delta=1+\gamma$ and f be an (α,β)-fuzzy

interior ideal of S. Then f_γ is an interior ideal of S.

Proof: The proof is similar to the proof of Theorem 3.6.

3.9. Theorem Let $2\delta = 1+\gamma$ and A be a non-empty subset of S. Then A is a subsemigroup of S if and only if the fuzzy subset f of S defined by

$$f(x) = \begin{cases} \geq \delta \text{ if } x \in A \\ \leq \gamma \quad \text{otherwise} \end{cases}$$

is an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S.

Proof: Let A be a subsemigroup of S.
(1) Let $x, y \in S$ and $t, r \in (\gamma, 1]$ be such that $x_t \in_\gamma f, y_r \in_\gamma f$. Then $f(x) \geq t > \gamma$ and $f(y) \geq r > \gamma$. Thus $x, y \in A$ and so $xy \in A$, that is $f(xy) \geq \delta$. If $\min\{t,r\} \leq \delta$, then $f(xy) \geq \delta \geq \min\{t,r\} > \gamma$. This implies $(xy)_{\min\{t,r\}} \in_\gamma f$. If $\min\{t,r\} > \delta$, then $f(xy) + \min\{t,r\} > \delta + \delta = 2\delta$. This implies $(xy)_{\min\{t,r\}} q_\delta f$. Hence f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S.
(2) Let $x, y \in S$ and $t, r \in (\gamma, 1]$ be such that $x_t q_\delta f, y_r q_\delta f$. Then $f(x) + t > 2\delta$ and $f(y) + r > 2\delta$. This implies $f(x) > 2\delta - t \geq 2\delta - 1 = \gamma$ and $f(y) > 2\delta - r \geq 2\delta - 1 = \gamma$. Thus $x, y \in A$ and so $xy \in A$. This implies $f(xy) \geq \delta$. Now if $\min\{t,r\} \leq \delta$, then $f(xy) \geq \delta \geq \min\{t,r\} > \gamma$, so $(xy)_{\min\{t,r\}} \in f$. If $\min\{t,r\} > \delta$, then $f(xy) + \min\{t,r\} > \delta + \delta = 2\delta$. Thus $(xy)_{\min\{t,r\}} q_\delta f$. Hence f is a $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S.
(3) Let $x, y \in S$ and $t, r \in (\gamma, 1]$ be such that $x_t \in_\gamma f$ and $y_r q_\delta f$. Then $f(x) \geq t > \gamma$ and $f(y) + r > 2\delta$. Thus $f(y) + r > 2\delta \Rightarrow f(y) > 2\delta - r \geq 2\delta - 1 = \gamma$. This implies $x, y \in A$ and so $xy \in A$. Analogous to (1) and (2) we obtain $(xy)_{\min\{t,r\}} \in_\gamma \vee q_\delta f$, that is f is an $(\in_\gamma \vee q_\delta, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S.

Conversely, assume that f is an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S. Then $A = f_\gamma$. It follows from Theorem 3.6 that A is a subsemigroup of S.

3.10. Corollary Let $2\delta = 1+\gamma$ and A be a non-empty subset of S. Then A is a subsemigroup of S if and only if χ_A, the characteristic function of A is an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S.
Similarly we can prove the following theorem.

3.11 Theorem Let $2\delta = 1+\gamma$ and A be a non-empty subset of S. Define a fuzzy subset f of S as

$$f(x) = \begin{cases} \geq \delta \text{ if } x \in A \\ \leq \gamma \quad \text{otherwise} \end{cases}$$

Then
(1) f is an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideal of S if and only if A is a left (right) ideal of S.
(2) f is an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal (bi-ideal) of S if and only if A is a generalized bi-ideal (bi-ideal) of S.
(3) f is an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of S if and only if A is an interior ideal of S.

3.12. Corollary (1) Let $2\delta = 1+\gamma$ and A be a non-empty subset of S. Then A is a left (right) ideal of S if and only if χ_A, the characteristic function of A is an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideal of S.
(2) Let $2\delta = 1+\gamma$ and A be a non-empty subset of S. Then A is a generalized bi-ideal (bi-ideal) of S if and only if χ_A, the characteristic function of A is an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal (bi-ideal) of S.
(3) Let $2\delta = 1+\gamma$ and A be a non-empty subset of S. Then A is an interior ideal of S if and only if χ_A, the characteristic function of A is an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of S.

It is easy to see that each (α, β)-fuzzy subsemigroup (left ideal, right ideal, generalized bi-ideal, bi-ideal, interior ideal) of S is an

$(\alpha, \in \vee q)$-fuzzy subsemigroup (left ideal, right ideal, generalized bi-ideal, bi-ideal, interior ideal) of S.

The following example shows that the converse is not true.

3.13 Example Consider the semigroup $S = \{a, b, c, d\}$

	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Define a fuzzy subset f of S as follows $f(a) = 0.5$, $f(b) = 0.4$, $f(c) = 0.6$ and $f(d) = 0.3$.

Thus

$$U(f;t) = \begin{cases} S & \text{if } 0 < t \le 0.3 \\ \{a,b,c\} & \text{if } 0.3 < t \le 0.4 \\ \{a,c\} & \text{if } 0.4 < t \le 0.5 \\ \{c\} & \text{if } 0.5 < t \le 0.6 \\ \varphi & \text{if } 0.6 < t \end{cases}$$

Then

(1) f is an $(\in_0, \in_0 \vee q_{0.4})$-fuzzy ideal of S.

(2) f is not an $(\in_0, \in_0)$-fuzzy ideal of S, because $c_{0.5} \in_0 f$ but $(cc)_{0.5} \overline{\in}_0 f$.

(3) f is not an $(\in_0, q_{0.4})$-fuzzy ideal of S, because $c_{0.25} \in_0 f$ but $(cc)_{0.25} \overline{q_{0.4}} f$.

3.14. Theorem (1) Every $(\in_\gamma \vee q_\delta, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S.

(2) Every $(\in_\gamma \vee q_\delta, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideal of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideal of S.

(3) Every $(\in_\gamma \vee q_\delta, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal (bi-ideal) of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal (bi-ideal) of S.

(4) Every $(\in_\gamma \vee q_\delta, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of S.

Proof: The proof follows from the fact that if $x_t \in_\gamma f$ then $x_t \in_\gamma \vee q_\delta f$.

3.15 Theorem (1) Every $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S.

(2) Every $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideal of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideal of S.

(3) Every $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal (bi-ideal) of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal (bi-ideal) of S.

(4) Every $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of S.

Proof: We prove only (1). Proofs of $(2), (3)$ and (4) are similar to the proof of (1).

Let f be a $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S. Let $x, y \in S$ and $t, r \in (\gamma, 1]$ be such that $x_t \in_\gamma f, y_r \in_\gamma f$. Then $f(x) \ge t > \gamma$ and $f(y) \ge r > \gamma$. Suppose $(xy)_{\min\{t,r\}} \overline{\in_\gamma \vee q_\delta}$ then $f(xy) < \min\{t, r\}$ and $f(xy) + \min\{t, r\} \le 2\delta \Rightarrow f(xy) < \delta$. Now $\max\{f(xy), \gamma\} < \min\{f(x), f(y), \delta\}$. Then select an $s \in (\gamma, 1]$ such that

$$2\delta - \max\{f(xy), \gamma\} > s \ge 2\delta - \min\{f(x), f(y), \delta\}$$
$$\Rightarrow 2\delta - f(xy) \ge 2\delta - \max\{f(xy), \gamma\}$$
$$> s \ge \max\{2\delta - f(x), 2\delta - f(y), \delta\}$$
$$\Rightarrow f(x) + s \ge 2\delta, \quad f(y) + s \ge 2\delta$$

and $f(xy) + s < 2\delta$ and $f(xy) < \delta < s$. Hence $x_s q_\delta f, y_s q_\delta f$ but $(xy)_s \overline{\in_\gamma \vee q_\delta} f$. This is a contradiction. Hence $(xy)_{\min\{t,r\}} \in_\gamma \vee q_\delta f$, that is f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S.

The above discussion shows that every (α, β)-fuzzy subsemigroup (left ideal, right ideal, generalized bi-ideal, bi-ideal, interior ideal) of a semigroup S is an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup (left ideal, right ideal, generalized bi-ideal, bi-ideal, interior ideal) of S. Also every $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup (left ideal, right ideal, generalized bi-ideal, bi-ideal, interior ideal) of a semigroup S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup (left ideal, right ideal, generalized bi-ideal, bi-ideal, interior ideal) of S. Thus in the theory of (α, β)-fuzzy subsemigroups (left ideals, right ideals, generalized bi-ideals, bi-ideals, interior ideals) of S, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroups (left ideals, right ideals, generalized bi-ideals, bi-ideals, interior ideals) play a central role.

4. $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals

We start this section with the following theorem.

4.1. Theorem For any fuzzy subset f of a semigroup S and for all $x, y, z \in S$ and $t, r \in (\gamma, 1]$ $(1a)$ is equivalent to $(1b)$, $(2a)$ is equivalent to $(2b)$, $(3a)$ is equivalent to $(3b)$ and $(4a)$ is equivalent to $(4b)$, where

$(1a)$ $x_t,\ y_r \in_\gamma f \Rightarrow (xy)_{\min\{t,r\}} \in_\gamma \vee q_\delta f$.

(1b) $\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), \delta\}$.

$(2a)$ $x_t \in_\gamma f \Rightarrow (yx)_t \in_\gamma \vee q_\delta f \quad ((xy)_t \in_\gamma \vee q_\delta f)$.

(2b) $\max\{f(yx), \gamma\} \geq \min\{f(x), \delta\} \quad (\max\{f(xy), \gamma\} \geq \min\{f(x), \delta\}$.

$(3a)$ $x_t,\ y_r \in_\gamma f \Rightarrow (xzy)_{\min\{t,r\}} \in_\gamma \vee q_\delta f$.

(3b) $\max\{f(xzy), \gamma\} \geq \min\{f(x), f(y), \delta\}$.

$(4a)$ $x_t \in_\gamma f \Rightarrow (yxz)_t \in_\gamma \vee q_\delta f$.

(4b) $\max\{f(yxz), \gamma\} \geq \min\{f(x), \delta\}$.

Proof: We prove only $(1a) \Leftrightarrow (1b)$. Proofs of the remaining parts are similar to this.

$(1a) \Rightarrow (1b)$ Let f be a fuzzy subset of S which satisfies $(1a)$. Let $x, y \in S$ be such that $\max\{f(xy), \gamma\} < \min\{f(x), f(y), \delta\}$. Select $t \in (\gamma, 1]$ such that $\max\{f(xy), \gamma\} < t \leq \min\{f(x), f(y), \delta\}$. Then $f(x) \geq t > \gamma$, $f(y) \geq t > \gamma$, $f(xy) < t$ and $f(xy) + t < \delta + \delta = 2\delta$, that is $x_t \in_\gamma f$, $y_t \in_\gamma f$ but $(xy)_t \overline{\in_\gamma \vee q_\delta} f$. Which is a contradiction. Hence $\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), \delta\}$.

$(1b) \Rightarrow (1a)$ Let f be a fuzzy subset of S which satisfies $(1b)$. Let $x, y \in S$ and $t, r \in (\gamma, 1]$ be such that $x_t \in_\gamma f$, $y_r \in_\gamma f$ but $(xy)_{\min\{t,r\}} \overline{\in_\gamma \vee q_\delta} f$. Then

$$f(x) \geq t > \gamma \quad (1)$$
$$f(y) \geq r > \gamma \quad (2)$$
$$f(xy) < \min\{t, r\} \quad (3)$$
and $$f(xy) + \min\{t, r\} \leq 2\delta \quad (4).$$

It follows from (3) and (4) that $f(xy) < \delta$. Now $\max\{f(xy), \gamma\} < \delta$ and $\max\{f(xy), \gamma\} < \min\{f(x), f(y)\}$. Thus $\max\{f(xy), \gamma\} < \min\{f(x), f(y), \delta\}$. Which is a contradiction. Hence $(xy)_{\min\{t,r\}} \in_\gamma \vee q_\delta f$.

From the above theorem we deduce that

4.2. Definition A fuzzy subset f of a semigroup S is called an

- $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S if it satisfies $(1b)$.
- $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideal of S if it satisfies $(2b)$.
- $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal of S if it satisfies $(3b)$.
- $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S if it satisfies $(1b)$ and $(3b)$.
- $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of S if it satisfies $(4b)$.

4.3. Definition Let f be a fuzzy subset of a semigroup S. We define

$f_r = \{x \in S : x_r \in_\gamma f\} = \{x \in S : f(x) \geq r > \gamma\} = U(f; r)$.

$f_r^\delta = \{x \in S : x_r q_\delta f\} = \{x \in S : f(x) + r > 2\delta\}$.

$[f]_r^\delta = \{x \in S : x_r \in_\gamma \vee q_\delta f\} = f_r \cup f_r^\delta$ for all $r \in (\gamma, 1]$.

4.4. Theorem Let f be a fuzzy subset of a semigroup S. Then

(1) f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S if and only if $U(f; t) (\neq \phi)$ is a subsemigroup of S for all $t \in (\gamma, \delta]$.

(2) f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideal of S if and only if $U(f; t) (\neq \phi)$ is a left (right) ideal of S for all $t \in (\gamma, \delta]$.

(3) f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal (bi-ideal) of S if and only if $U(f;t)(\neq \phi)$ is a generalized bi-ideal (bi-ideal) of S for all $t \in (\gamma, \delta]$.

(4) f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of S if and only if $U(f;t)(\neq \phi)$ is a interior ideal of S for all $t \in (\gamma, \delta]$.

Proof: (1) Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy subsemigroup of S and $x, y \in U(f;t)$ for some $t \in (\gamma, \delta]$. Then $f(x) \geq t$ and $f(y) \geq t$. By hypothesis $\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), \delta\} \geq \min\{t, \delta\} = t \Rightarrow f(xy) \geq t$. Hence $xy \in U(f;t)$, that is $U(f;t)$ is a subsemigroup of S.

Conversely, assume that $U(f;t) \neq \phi$ is a subsemigroup of S for all $t \in (\gamma, \delta]$. Suppose that there exist $x, y \in S$ such that $\max\{f(xy), \gamma\} < \min\{f(x), f(y), \delta\}$. Choose $t \in (\gamma, \delta]$ such that $\max\{f(xy), \gamma\} < t \leq \min\{f(x), f(y), \delta\}$. This implies $f(x) \geq t$, $f(y) \geq t$ and $f(xy) < t$, that is $x, y \in U(f;t)$ but $xy \notin U(f;t)$ which is a contradiction. Hence $\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), \delta\}$, that is f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S.

Similarly we can prove (2), (3) and (4).

From the above Theorem it follows that

(1) Every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of a semigroup S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of S.

(2) Every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideal of a semigroup S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal ideal of S.

(3) Every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of a semigroup S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal of S.

4.5. Theorem Let f be a fuzzy subset of a semigroup S and $2\delta = 1 + \gamma$. Then

(1) f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S if and only if $f_r^\delta(\neq \phi)$ is a subsemigroup of S for all $r \in (\gamma, \delta]$

(2) f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left(right) ideal of S if and only if $f_r^\delta(\neq \phi)$ is a left(right) ideal of S for all $r \in (\gamma, \delta]$

(3) f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal (bi-ideal) of S if and only if $f_r^\delta(\neq \phi)$ is a generalized bi-ideal (bi-ideal) of S for all $r \in (\gamma, \delta]$

(4) f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of S if and only if $f_r^\delta(\neq \phi)$ is an interior ideal of S for all $r \in (\gamma, \delta]$

Proof: (1) Suppose f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S and $x, y \in f_r^\delta$. Then $x_r, y_r q_\delta f$, that is $f(x) + r > 2\delta$ and $f(y) + r > 2\delta \Rightarrow f(x) > 2\delta - r \geq 2\delta - 1 = \gamma$ and similarly $f(y) > \gamma$. By hypothesis

$$\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), \delta\}$$
$$\Rightarrow f(xy) \geq \min\{f(x), f(y), \delta\}$$
$$\Rightarrow f(xy) > \min\{2\delta - r, 2\delta - r, \delta\}.$$

Since $r \in (\gamma, \delta]$, $\delta < r \leq 1 \Rightarrow 2\delta - r < \delta$. Thus $f(xy) > 2\delta - r \Rightarrow f(xy) + r > 2\delta \Rightarrow xy \in f_r^\delta$. Hence f_r^δ is a subsemigroup of S.

Conversely, assume that $f_r^\delta(\neq \phi)$ is a subsemigroup of S for all $r \in (\delta, 1]$. Let $x, y \in S$ be such that $\max\{f(xy), \gamma\} < \min\{f(x), f(y), \delta\} \Rightarrow$
$2\delta - \min\{f(x), f(y), \delta\} < 2\delta - \max\{f(xy), \gamma\}$
$\Rightarrow \max\{2\delta - f(x), 2\delta - f(y), \delta\} < \min\{2\delta - f(xy), 2\delta - \gamma\}$.
Take $r \in (\gamma, \delta]$ such that $\max\{2\delta - f(x), 2\delta - f(y), \delta\} < r \leq \min\{2\delta - f(xy), 2\delta - \gamma\}$.
Then $2\delta - f(x) < r, 2\delta - f(y) < r$ and $r \leq 2\delta - f(xy) \Rightarrow$ $f(x) + r > 2\delta$ and $f(y) + r > 2\delta$ but $f(xy) + r \leq 2\delta$, that is $x_r q_\delta f, y_r q_\delta f$ but $(xy)_r \overline{q}_\delta f$. Which is a contradiction. Hence $\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), \delta\}$.

Similarly, we can prove the parts (2), (3) and (4).

4.6. Theorem Let f be a fuzzy subset of a semigroup S and $2\delta = 1 + \gamma$. Then

(1) f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S if and only if $[f]_r^\delta(\neq \phi)$ is a subsemigroup of

S for all $r \in (\gamma, 1]$.

(2) f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left(right) ideal of S if and only if $[f]_r^\delta (\neq \phi)$ is a left(right) ideal of S for all $r \in (\gamma, 1]$.

(3) f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal (bi-ideal) of S if and only if $[f]_r^\delta (\neq \phi)$ is a generalized bi-ideal (bi-ideal) of S for all $r \in (\gamma, 1]$.

(4) f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy interior ideal of S if and only if $[f]_r^\delta (\neq \phi)$ is an interior ideal of S for all $r \in (\gamma, 1]$.

Proof: (1) Suppose f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S and $x, y \in [f]_r^\delta$. Then $x_r \in_\gamma \vee q_\delta f$ and $y_r \in_\gamma \vee q_\delta f$, that is $f(x) \geq r > \gamma$ or $f(x) + r > 2\delta$ and $f(y) \geq r > \gamma$ or $f(y) + r > 2\delta$. Thus $f(x) \geq r > \gamma$ or $f(x) > 2\delta - r \geq 2\delta - 1 = \gamma$ and $f(y) \geq r > \gamma$ or $f(y) > 2\delta - r \geq 2\delta - 1 = \gamma$. If $r \in (\gamma, \delta]$, then $\gamma < r \leq \delta$. This implies $2\delta - r \geq \delta \geq r$. Then it follows from the above that $f(x) \geq r$ and $f(y) \geq r$. By hypothesis $\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), \delta\}$

$\Rightarrow f(xy) \geq \min\{f(x), f(y), \delta\} \geq \min\{r, r, r\} = r$

and so $(xy)_r \in_\gamma f$. Thus $xy \in [f]_r^\delta$.

If $r \in (\delta, 1]$, then $\delta < r \leq 1$. This implies $2\delta - r < \delta < r$. Then it follows that $f(x) > 2\delta - r$ and $f(y) > 2\delta - r$.

Now by hypothesis $\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), \delta\}$

$\Rightarrow f(xy) \geq \min\{f(x), f(y), \delta\} >$

$\min\{2\delta - r, 2\delta - r, 2\delta - r\} = 2\delta - r$

$\Rightarrow f(xy) + r > 2\delta \Rightarrow (xy)_r q_\delta f$.

This implies $xy \in [f]_r^\delta$. Thus $[f]_r^\delta$ is a subsemigroup of S.

Conversely, assume that $[f]_r^\delta$ is a subsemigroup of S for all $r \in (\gamma, 1]$. Let $x, y \in S$ be such that $\max\{f(xy), \gamma\} < \min\{f(x), f(y), \delta\}$. Select $r \in (\gamma, 1]$ such that $\max\{f(xy), \gamma\} < r \leq \min\{f(x), f(y), \delta\}$. Then $x_r \in_\gamma f, y_r \in_\gamma f$ but $(xy)_r \overline{\in_\gamma \vee q_\delta} f$. Which contradicts our hypothesis. Hence $\max\{f(xy), \gamma\} \geq \min\{f(x), f(y), \delta\}$, that is f is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S.

Similarly, we can prove the parts (2), (3) and (4).

4.7. Theorem (1) The intersection of any family of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroups of S is again an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S.

(2) The intersection of any family of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideals of S is again an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideal of S.

(3) The intersection of any family of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal (bi-ideal) of S is again an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal (bi-ideal) of S.

(4) The intersection of any family of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideals of S is again an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of S.

Proof: Straightforward.

4.8. Theorem The union of any family of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideals of S is again an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideal of S.

Proof: Straightforward.

4.9. Proposition Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S and g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal of S. Then fg is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of S.

Proof: Straightforward.

Next we show that if f and g are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals of S, then $fg \not\leq f \wedge g$.

4.10. Example Let $S = \{a, b, c, d\}$ be a semigroup with the following multiplication table

	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Define fuzzy subset f, g of S by $f(a)=0.6$, $f(b)=0.3$, $f(c)=0.4$, $f(d)=0.1$, $g(a)=0.65$, $g(b)=0.3$, $g(c)=0.4$, $g(d)=0.2$.
Then

$$U(f;t)=\begin{cases}\{a,b,c,d\} \text{ if } 0<t\le 0.1\\ \{a,b,c\} \text{ if } 0.1<t\le 0.3\\ \{a,c\} \text{ if } 0.3<t\le 0.4\\ \{a\} \text{ if } 0.4<t\le 0.6\\ \phi \text{ if } 0.6<t\end{cases}$$

$$U(g;t)=\begin{cases}\{a,b,c,d\} \text{ if } 0<t\le 0.2\\ \{a,b,c\} \text{ if } 0.2<t\le 0.3\\ \{a,c\} \text{ if } 0.3<t\le 0.4\\ \{a\} \text{ if } 0.4<t\le 0.65\\ \phi \text{ if } 0.65<t\end{cases}$$

By Theorem 4.4, f and g are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals of S for $\gamma=0$ and $\delta=0.3$. But $fg(b)=\underset{b=xy}{\vee}\{f(x)\wedge g(y)\}=\vee\{0.4, 0.1, 0.1\}=0.4\not\le(f\wedge g)(b)=0.3.$
Hence $fg\not\le f\wedge g$ in general.

4.11. Definition Let f, g be fuzzy subsets of a semigroup S. We define the fuzzy subsets $f^*, f\wedge^* g, f\vee^* g$ and $f*g$ of S as follows:

$f^*(x)=(f(x)\vee\gamma)\wedge\delta$

$(f\wedge^* g)(x)=(((f\wedge g)(x))\vee\gamma)\wedge\delta$

$(f\vee^* g)(x)=(((f\vee g)(x))\vee\gamma)\wedge\delta$

$(f*g)(x)=(((fg)(x))\vee\gamma)\wedge\delta$

for all $x\in S$.

4.12. Lemma Let f, g be fuzzy subsets of a semigroup S. Then the following hold:

(1) $f\wedge^* g=f^*\wedge g^*$

(2) $f\vee^* g=f^*\vee g^*$

(3) $f*g\ge f^*g^*$.

Proof: Proofs of (1) and (2) are straightforward.
(3) Let $x\in S$. If x is not expressible as $x=yz$ for all $y,z\in S$, then $(f*g)(x)=(((fg)(x))\vee\gamma)\wedge\delta=\gamma\wedge\delta\ge 0=f^*g^*(x)$.
Otherwise

$$(f*g)(x)=((fg(x))\vee\gamma)\wedge\delta=\left(\left(\underset{x=yz}{\vee}fg(x)\right)\vee\gamma\right)\wedge\delta$$
$$=\left(\underset{x=yz}{\vee}((f(y)\vee\gamma)\wedge((g(z)\vee\gamma)))\right)\wedge\delta$$
$$=\left(\underset{x=yz}{\vee}([(f(y)\vee\gamma)\wedge\delta]\wedge([(g(z)\vee\gamma)\wedge\delta])\right)$$
$$=\underset{x=yz}{\vee}\left(f^*(y)\wedge g^*(z)\right)=\left(f^*g^*\right)(x)$$

4.13. Lemma Let A, B be non-empty subsets of a semigroup S. Then the following hold.

(1) $\overset{*}{\chi_A}\wedge\overset{*}{\chi_B}=\overset{*}{\chi}_{A\cap B}$.

(2) $\chi_A*\chi_B=\overset{*}{\chi}_{AB}$.

4.14. Theorem (1) A fuzzy subset f of a semigroup S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S if and only if $f*f\le f^*$.
(2) A fuzzy subset f of a semigroup S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal of S if and only if $f*\mathbf{S}*f\le f^*$.
(3) A fuzzy subset f of a semigroup S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$ fuzzy left (right) ideal of S if and only if $\mathbf{S}*f\le f^*$. ($f*\mathbf{S}\le f^*$).
(4) A fuzzy subset f of a semigroup S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S if and only if $f*f\le f^*$ and $f*\mathbf{S}*f\le f^*$.
Where $\mathbf{S}$ is the fuzzy subset of S mapping every element of S on 1.

Proof: Straightforward.

4.15. Definition A fuzzy subset f of a semigroup S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal of S if and only if $(f*\mathbf{S})\wedge(\mathbf{S}*f)\le f^*$.

4.16. Lemma A non-empty subset A of a semigroup S is a quasi-ideal of S if and only if the characteristic function of A is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal of S.
Proof: Straightforward.

5. Regular Semigroups

Recall that a semigroup S is regular if for each

$x \in S$ there exists $a \in S$ such that $x = xax$.

5.1. Proposition In a regular semigroup S, every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of S.

Proof: Straightforward.

5.2. Proposition In a regular semigroup S, every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S.

Proof: Straightforward.

5.3. Theorem The following assertions are equivalent for a semigroup S:

(1) S is regular.

(2) $f \overset{*}{\wedge} g = f * g$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal f and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal g of S.

Proof: $(1) \Rightarrow (2)$ Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal and g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S. Then by Lemma 4.12, $f * g \le f \overset{*}{\wedge} g$.

Let $x \in S$. Then there exists $a \in S$ such that $x = xax$. Now

$$(f * g)(x) = \left(\left(\bigvee_{x=yz} (f(y) \wedge g(z))\right) \vee \gamma\right) \wedge \delta$$
$$\ge \left((f(x) \wedge g(ax)) \vee \gamma\right) \wedge \delta$$
$$= \left((f(x) \wedge (g(ax) \vee \gamma)) \vee \gamma\right) \wedge \delta$$
$$\ge \left((f(x) \wedge (g(x) \wedge \delta)) \vee \gamma\right) \wedge \delta$$
$$= \left((f(x) \wedge g(x)) \vee \gamma\right) \wedge \delta$$
$$= \left(f \overset{*}{\wedge} g\right)(x).$$

Thus $f * g \ge f \overset{*}{\wedge} g$. Hence $f * g = f \overset{*}{\wedge} g$.

$(2) \Rightarrow (1)$ Let R be a right ideal and L a left ideal of S. Then χ_R and χ_L are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideals of S, respectively. By hypothesis $\chi_R * \chi_L = \chi_R \overset{*}{\wedge} \chi_L$. By Lemma 4.13, this implies that $\overset{*}{\chi}_{RL} = \overset{*}{\chi}_{R \cap L} \Rightarrow RL = R \cap L$. Hence S is a regular semigroup.

5.4. Theorem The following assertions are equivalent for a semigroup S:

(1) S is regular.

(2) $f \overset{*}{\wedge} g \overset{*}{\wedge} h \le f * g * h$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal f, every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal g and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal h of S.

(3) $f \overset{*}{\wedge} g \overset{*}{\wedge} h \le f * g * h$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal f, every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal g and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal h of S.

(4) $f \overset{*}{\wedge} g \overset{*}{\wedge} h \le f * g * h$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal f, every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal g and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal h of S.

Proof: $(1) \Rightarrow (2)$ Let f, g, h be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S, respectively. Let $x \in S$. Then there exists $a \in S$ such that $x = xax$. Thus we have

$$(f * g * h)(x) = \left[\left(\left(\bigvee_{x=yz} \left(f(y) \wedge (g * h)(z)\right)\right) \vee \gamma\right) \wedge \delta\right]$$
$$\ge \left[\left(\left(f(xa) \wedge (g * h)(x)\right) \vee \gamma\right) \wedge \delta\right]$$
$$= \left[\left(\left((f(xa) \vee \gamma) \wedge (g * h)(x)\right) \vee \gamma\right) \wedge \delta\right]$$
$$\ge \left[\left(\left((f(x) \wedge \delta) \wedge (g * h)(x)\right) \vee \gamma\right) \wedge \delta\right]$$
$$= \left[\left(\left(f(x) \wedge \left(\left(\bigvee_{x=yz} (g(y) \wedge h(z))\right) \vee \gamma\right) \wedge \delta\right) \vee \gamma\right) \wedge \delta\right]$$
$$\ge \left[\left(\left(f(x) \wedge \left((g(x) \wedge h(ax)) \vee \gamma\right) \wedge \delta\right) \vee \gamma\right) \wedge \delta\right]$$

$$=\left[\left(\left(f(x)\wedge\left(\left(g(x)\wedge\left(h(ax)\vee\gamma\right)\right)\right)\wedge\delta\right)\vee\gamma\right)\wedge\delta\right]$$
$$\geq\left[\left(\left(f(x)\wedge\left(\left(g(x)\wedge\left(h(x)\wedge\delta\right)\right)\right)\wedge\delta\right)\vee\gamma\right)\wedge\delta\right]$$
$$=\left[\left(\left(f(x)\wedge g(x)\wedge h(x)\right)\vee\gamma\right)\wedge\delta\right]$$
$$=\left(f\overset{*}{\wedge}g\overset{*}{\wedge}h\right)(x).$$

Thus $f\overset{*}{\wedge}g\overset{*}{\wedge}h\leq f*g*h$.

$(2)\Rightarrow(3)\Rightarrow(4)$ are straightforward.

$(4)\Rightarrow(1)$ Let f be an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy right ideal and g be an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy left ideal of S. Since $\mathbf{S}$ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$ -fuzzy quasi-ideal of S, so by hypothesis we have $\left(f\overset{*}{\wedge}g\right)(x)=\left(f\overset{*}{\wedge}\mathbf{S}\overset{*}{\wedge}g\right)(x)\leq(f*\mathbf{S}*g)(x)\leq(f*g)(x)$.

But $\left(f\overset{*}{\wedge}g\right)(x)\geq(f*g)(x)$ always holds.

Thus $f\overset{*}{\wedge}g=f*g$. Hence by Theorem 5.3, S is regular.

5.5. Theorem The following assertions are equivalent for a semigroup S:

(1) S is regular.

(2) $\overset{*}{f}=f*\mathbf{S}*f$ for every $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy generalized bi-ideal f of S.

(3) $\overset{*}{f}=f*\mathbf{S}*f$ for every $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy bi-ideal f of S.

(4) $\overset{*}{f}=f*\mathbf{S}*f$ for every $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy quasi-ideal f of S.

Proof: The proof is similar to the proof of Theorem 5.4.

5.6. Theorem The following assertions are equivalent for a semigroup S:

(1) S is regular.

(2) $f\overset{*}{\wedge}g=f*g*f$ for every $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy quasi-ideal f and every $(\in_\gamma,\in_\gamma\vee q_\delta)$ -fuzzy ideal g of S.

(3) $f\overset{*}{\wedge}g=f*g*f$ for every $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy quasi-ideal f and every $(\in_\gamma,\in_\gamma\vee q_\delta)$ -fuzzy interior ideal g of S.

(4) $f\overset{*}{\wedge}g=f*g*f$ for every $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy bi-ideal f and every $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy ideal g of S.

(5) $f\overset{*}{\wedge}g=f*g*f$ for every $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy bi-ideal f and every $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy interior ideal g of S.

(6) $f\overset{*}{\wedge}g=f*g*f$ for every $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy generalized bi-ideal f and every $(\in_\gamma,\in_\gamma\vee q_\delta)$ -fuzzy ideal g of S.

(7) $f\overset{*}{\wedge}g=f*g*f$ for every $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy generalized bi-ideal f and every $(\in_\gamma,\in_\gamma\vee q_\delta)$ -fuzzy interior ideal g of S.

Proof: The proof is similar to the proof of Theorem 5.4.

5.7. Theorem The following assertions are equivalent for a semigroup S:

(1) S is regular.

(2) $f\overset{*}{\wedge}g\leq f*g$ for every $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy quasi-ideal f and every $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy left ideal g of S.

(3) $f\overset{*}{\wedge}g\leq f*g$ for every $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy bi-ideal f and every $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy left ideal g of S.

(4) $f\overset{*}{\wedge}g\leq f*g$ for every $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy generalized bi-ideal f and every $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy left ideal g of S.

Proof: The proof is similar to the proof of Theorem 5.4.

6. Intra regular Semigroups

A semigroup S is said to be intra regular if for each $x\in S$ there exist $a,b\in S$ such that

$x = axxb$. In general, neither regular semigroup is intra regular nor is intra regular semigroup regular. If S is commutative then both the concepts coincide.

6.1. Example Let A be a countably infinite set and $S = \{\alpha : A \to A : \alpha \text{ is one one and } A - \alpha(A) \text{ is infinite}\}$

Then S is a semigroup with respect to the composition of functions and is called Baer-Levi Semigroup (cf. 24). This semigroup is right cancellative, right simple without idempotents (cf. [24, Th. 8.2). Thus S is not regular but intra regular.

6.2. Example Consider the semigroup $S = \{0,1,2,3,4\}$.

	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	1	2
2	0	1	2	0	0
3	0	0	0	3	4
4	0	3	4	0	0

This semigroup S is regular but not intra regular.

6.3. Theorem The following assertions are equivalent for a semigroup S :

(1) S is intra regular.

(2) $f \overset{*}{\wedge} g \le f * g$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal f and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal g of S.

Proof: $(1) \Rightarrow (2)$ Let f be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal and g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal of S. Let $x \in S$. Then there exist $a, b \in S$ such that $x = axxb$. Now

$$(f * g)(x) = \left(\left(\bigvee_{x=yz}\left(f(y) \wedge g(z)\right)\right) \vee \gamma\right) \wedge \delta$$
$$\ge \left(\left(f(ax) \wedge g(xb)\right) \vee \gamma\right) \wedge \delta$$
$$= \left(\left(\left(f(ax) \vee \gamma\right) \wedge \left(g(xb) \vee \gamma\right)\right) \vee \gamma\right) \wedge \delta$$
$$\ge \left(\left(\left(f(x) \wedge \delta\right) \wedge \left(g(x) \wedge \delta\right)\right) \vee \gamma\right) \wedge \delta$$
$$= \left(\left(f(x) \wedge g(x)\right) \vee \gamma\right) \wedge \delta$$
$$= \left(f \overset{*}{\wedge} g\right)(x).$$

Thus $f * g \ge f \overset{*}{\wedge} g$.

$(2) \Rightarrow (1)$ Let R be any right ideal and L be any left ideal of S. Then χ_R and χ_L are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideals of S, respectively. By hypothesis $\chi_L * \chi_R \ge \chi_L \overset{*}{\wedge} \chi_R$. By Lemma 4.13, this implies that $\overset{*}{\chi}_{LR} \ge \overset{*}{\chi}_{L\cap R} \Rightarrow LR \supseteq L \cap R$. Hence S is an intra regular semigroup.

6.4. Theorem The following assertions are equivalent for a semigroup S :

(1) S is both regular and intra regular.

(2) $\overset{*}{f} = f * f$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal f of S.

(3) $\overset{*}{f} = f * f$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal f of S.

(4) $f \overset{*}{\wedge} g \le f * g$ for all $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideals f, g of S.

(5) $f \overset{*}{\wedge} g \le f * g$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal f and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal g of S.

(6) $f \overset{*}{\wedge} g \le f * g$ for all $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals f, g of S.

Proof: $(1) \Rightarrow (6)$ Let f, g be $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals of S. Let $x \in S$. Then there exist $a, b, c \in S$ such that $x = axxb$ and $x = xcx$. Thus
$x = xcx = xcxcx = xc(axxb)cx = (xcax)(xbcx)$.
Thus we have

$$(f * g)(x) = \left(\left(\bigvee_{x=yz}\left(f(y) \wedge g(z)\right)\right) \vee \gamma\right) \wedge \delta$$
$$\ge \left(\left(f(xcax) \wedge g(xbcx)\right) \vee \gamma\right) \wedge \delta$$
$$= \left(\left(\left(f(xcax) \vee \gamma\right) \wedge \left(g(xbcx) \vee \gamma\right)\right) \vee \gamma\right) \wedge \delta$$

$\geq \left(\left(\left(f(x) \wedge \delta \right) \wedge \left(g(x) \wedge \delta \right) \right) \vee \gamma \right) \wedge \delta$

$= \left(\left(f(x) \wedge g(x) \right) \vee \gamma \right) \wedge \delta$

$= \left(f \overset{*}{\wedge} g \right)(x).$

Thus $f * g \geq f \overset{*}{\wedge} g$.

$(6) \Rightarrow (5) \Rightarrow (4)$ are obvious.

$(4) \Rightarrow (2)$ Take $f = g$ in (4), we get $f * f \geq \overset{*}{f}$. Since every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S, $f * f \leq \overset{*}{f}$. Thus $f * f = \overset{*}{f}$.

$(6) \Rightarrow (3)$ Take $f = g$ in (6), we get $f * f \geq \overset{*}{f}$. Since every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S, so $f * f \leq \overset{*}{f}$. Thus $f * f = \overset{*}{f}$.

$(3) \Rightarrow (2)$ Obvious.

$(2) \Rightarrow (1)$ Let Q be a quasi-ideal of S. Then χ_Q is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal of S. Hence by hypothesis $\chi_Q * \chi_Q = \overset{*}{\chi}_Q$. This implies that $\overset{*}{\chi}_{QQ} = \overset{*}{\chi}_Q$, that is $QQ = Q$. Hence by Theorem 2.3, S is both regular and intra regular.

6.5. Theorem The following assertions are equivalent for a semigroup S:

(1) S is both regular and intra regular.

(2) $f \overset{*}{\wedge} g \leq (f * g) \wedge (g * f)$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal f and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal g of S.

(3) $f \overset{*}{\wedge} g \leq (f * g) \wedge (g * f)$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal f and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal g of S.

(4) $f \overset{*}{\wedge} g \leq (f * g) \wedge (g * f)$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal f and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal g of S.

(5) $f \overset{*}{\wedge} g \leq (f * g) \wedge (g * f)$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal f and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal g of S.

(6) $f \overset{*}{\wedge} g \leq (f * g) \wedge (g * f)$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal f and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal g of S.

(7) $f \overset{*}{\wedge} g \leq (f * g) \wedge (g * f)$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal f and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal g of S.

(8) $f \overset{*}{\wedge} g \leq (f * g) \wedge (g * f)$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal f and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal g of S.

(9) $f \overset{*}{\wedge} g \leq (f * g) \wedge (g * f)$ for all $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal f, g of S.

(10) $f \overset{*}{\wedge} g \leq (f * g) \wedge (g * f)$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal f and every $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy bi-ideal g of S.

(11) $f \overset{*}{\wedge} g \leq (f * g) \wedge (g * f)$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal f and every $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy generalized bi-ideal g of S.

(12) $f \overset{*}{\wedge} g \leq (f * g) \wedge (g * f)$ for all $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals f, g of S.

(13) $f \overset{*}{\wedge} g \leq (f * g) \wedge (g * f)$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal f and every $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy generalized bi-ideal g of S.

(14) $f \overset{*}{\wedge} g \leq (f * g) \wedge (g * f)$ for all

$(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal f, g of S.

Proof: The proof is similar to the proof of Theorem 6.4.

7. Semisimple Semigroups

Recall that a semigroup S is semisimple if every two sided ideal of S is idempotent. It is clear that a semigroup S is semisimple if and only if $a \in (SaS)(SaS)$ for every $a \in S$, that is there exist $x, y, z, t \in S$ such that $a = (xay)(taz)$.

7.1. Theorem In a semisimple semigroup S, a fuzzy subset f of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of S if and only if it is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of S.

Proof: Straightforward.

7.2. Theorem For a semigroup S the following assertions are equivalent

(1) S is semisimple.

(2) $f * f = \overset{*}{f}$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal f of S.

(3) $f * f = \overset{*}{f}$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal f of S.

(4) $f \overset{*}{\wedge} g = f * g$ for all $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals f, g of S.

(5) $f \overset{*}{\wedge} g = f * g$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal f and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal g of S.

(6) $f \overset{*}{\wedge} g = f * g$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal f and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal g of S.

(7) $f \overset{*}{\wedge} g = f * g$ for all $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideals f, g of S.

Proof: $(1) \Rightarrow (7)$ Let S be a semisimple semigroup and f, g be $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideals of S. Let $x \in S$. Then there exist $a, b, c, d \in S$ such that $x = (axb)(cxd)$. Thus we have

$$
\begin{aligned}
(f * g)(x) &= \left(\left(\bigvee_{x=yz} (f(y) \wedge g(z))\right) \vee \gamma\right) \wedge \delta \\
&\geq ((f(axb) \wedge g(cxd)) \vee \gamma) \wedge \delta \\
&= (((f(axb) \vee \gamma) \wedge (g(cxd) \vee \gamma)) \vee \gamma) \wedge \delta \\
&\geq (((f(x) \wedge \delta) \wedge (g(x) \wedge \delta)) \vee \gamma) \wedge \delta \\
&= ((f(x) \wedge g(x)) \vee \gamma) \wedge \delta \\
&= \left(f \overset{*}{\wedge} g\right)(x).
\end{aligned}
$$

Thus $f * g \geq f \overset{*}{\wedge} g$. Since every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of S in a semisimple semigroup is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of S, so $f * g \leq f \overset{*}{\wedge} g$. Hence $f * g = f \overset{*}{\wedge} g$.

$(7) \Rightarrow (6) \Rightarrow (4) \Rightarrow (2)$, $(7) \Rightarrow (3) \Rightarrow (2)$ and $(7) \Rightarrow (5) \Rightarrow (4)$ are obvious.

$(2) \Rightarrow (1)$ Let A be any ideal of S. Then χ_A is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of S. Thus by hypothesis $\chi_A * \chi_A = \overset{*}{\chi}_A$, that is $AA = A$. Hence S is a semisimple semigroup.

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