

Soliton solutions to a few coupled nonlinear wave equations by tanh method

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Abstract

In this paper, tanh method is applied to obtain exact solutions for two systems of nonlinear wave equations, namely, two component evolutionary system of homogeneous KdV equations of order 3 (type I as well as type II). Moreover, traveling wave hypothesis is used to obtain sech solution of type II coupled KdV system, in a more general setting. The results show that this method presents exact solutions compared with other methods and it is a powerful tool for solving systems of nonlinear PDEs.

Keywords: Tanh method; nonlinear system of PDES; exact Solutions

1. Introduction

Nonlinear coupled partial differential equations are very important in a variety of scientific fields, especially in fluid mechanics, solid state physics, plasma physics, plasma waves, capillary-gravity waves and chemical physics. The nonlinear wave phenomena observed in the above mentioned scientific fields are often modeled by the bell-shaped sech solutions and the kink-shaped tanh solutions. The availability of these exact solutions for those nonlinear equations can greatly facilitate the verification of numerical solvers on the stability analysis of the solution [1], [2]. The hyperbolic tangent (tanh) method is a powerful technique to symbolically compute traveling wave solutions of one-dimensional nonlinear wave and evolution equations. In particular, the method is well suited for problems where dispersion, convection, and reaction diffusion phenomena play an important role [3]. In this paper, we solve three systems of nonlinear wave equations; these systems can be seen in [4]. In mathematical physics, they play a major role in various fields, such as plasma physics, fluid mechanics, optical fibers, solid state physics, geochemistry, and so on.

These nonlinear of those systems are called component evolutionary systems of homogeneous KdV equations of order 3 (type I and type II) respectively given by

$$\begin{aligned}u_t - u_{xxx} - uu_x - vv_x &= 0, \\v_t + 2v_{xxx} + uv_x &= 0,\end{aligned}\tag{1}$$

and

$$\begin{aligned}u_t - u_{xxx} - 2vu_x - uv_x &= 0, \\v_t - uu_x &= 0.\end{aligned}\tag{2}$$

2. Outline of the tanh method

The tanh method will be introduced as presented by Malfliet [5] and by Wazwaz [6], [7], [8]. The tanh method is based on a priori assumption that the traveling wave solutions can be expressed in terms of the tanh function to solve the coupled KdV equations. The tanh method is developed by Malfliet [5]. The method is applied to find out exact solutions of a coupled system of nonlinear differential equations with three unknowns:

$$\begin{aligned}P_1(u, v, u_x, v_x, u_t, v_t, u_{xx}, v_{xx}, \dots) &= 0 \\P_2(u, v, u_x, v_x, u_t, v_t, u_{xx}, v_{xx}, \dots) &= 0\end{aligned}\tag{3}$$

where P_1 , and P_2 are polynomials of the variables u, v and their derivatives. We consider $u(x, t) = u(\xi)$ and $v(x, t) = v(\xi)$ where $\xi = kx - \lambda t$, and use the following changes:

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$$\frac{\partial}{\partial t} = -\lambda \frac{d}{d\xi}, \quad \frac{\partial}{\partial x} = k \frac{d}{d\xi},$$

$$\frac{\partial^2}{\partial x^2} = k^2 \frac{d^2}{d\xi^2}, \quad \frac{\partial^3}{\partial x^3} = k^3 \frac{d^3}{d\xi^3},$$

and so on, then (3) become ordinary differential equations:

$$Q_1(U, U', U'', U''', \dots, V, V', V'', V''', \dots) = 0 \quad (4)$$

$$Q_2(U, U', U'', U''', \dots, V, V', V'', V''', \dots) = 0$$

with Q_1, Q_2 being another polynomials form of their argument, which will be called the reduced ordinary differential equations (4). Integrating (4), as long as all terms contain derivatives, the integration constants are considered to be zeros in view of the localized solutions. However, the nonzero constants can be used and handled as well [8]. Now, finding the traveling wave solutions to (3) is equivalent to obtaining the solution to the reduced ordinary differential equations (4). For the tanh method, we introduce the new independent variable

$$Y(x, t) = \tanh(\xi) \quad (5)$$

that leads to the change of variables:

$$\frac{d}{d\xi} = (1 - Y^2) \frac{d}{dY},$$

$$\frac{d^2}{d\xi^2} = -2Y(1 - Y^2) \frac{d}{dY} + (1 - Y^2)^2 \frac{d^2}{dY^2}, \quad (6)$$

$$\frac{d^3}{d\xi^3} = 2(1 - Y^2)(3Y^2 - 1) \frac{d}{dY}$$

$$- 6Y(1 - Y^2)^2 \frac{d^2}{dY^2} + (1 - Y^2)^3 \frac{d^3}{dY^3}.$$

The next crucial step is that the solution we are looking for is expressed in the form

$$u(x, t) = U(\xi) = \sum_{i=1}^m a_i Y^i \quad (7)$$

$$v(x, t) = V(\xi) = \sum_{i=1}^n b_i Y^i$$

where the parameters m , and n can be found by balancing the highest-order linear term with the nonlinear terms in (3), and k , λ , a_0, a_1, \dots, a_m , b_0, b_1, \dots, b_n are to be determined. Substituting (7)

into (4) and equating the coefficients of Y^i to zero, yields to the set of algebraic equations for k , λ , a_0, a_1, \dots, a_m , b_0, b_1, \dots, b_n . Having determined these parameters, knowing that m , and n are positive integers in most cases, and using (7), we obtain the expressions for $u(x, t)$ and $v(x, t)$ in a closed form [8]. The tanh method seems to be a powerful tool in dealing with coupled nonlinear physical models.

3. Applications

In this section, we apply the tanh method to the proposed systems. These systems were studied by [9] by applying HPM.

Example 1. Consider a two component evolutionary system of KdV equations of order 3 (type I) [9]:

$$u_t - u_{xxx} - uu_x - vv_x = 0, \quad (8)$$

$$v_t + 2v_{xxx} + uv_x = 0.$$

Using the traveling wave transformations:

$$u(x, t) = U(\xi) = \sum_{i=1}^m a_i Y^i, \quad (9)$$

$$v(x, t) = V(\xi) = \sum_{i=1}^n b_i Y^i$$

where

$$Y = \tanh(\xi) \quad (10)$$

$$\xi = kx - \lambda t \quad (11)$$

the nonlinear system of partial differential equations (8) is carried to a system of ordinary differential equations

$$-\lambda U' - k^3 U''' - kUU' - kVV' = 0, \quad (12)$$

$$-\lambda V' + 2k^3 V''' + kUV' = 0.$$

By postulating tanh series, and using the transformations given by (10) and (11), the first equation in (12) reduces to

$$\begin{aligned}
 &-\lambda(1-Y^2)\frac{dU}{dY}-k^3[2(1-Y^2)(3Y^2-1)\frac{dU}{dY} \\
 &-6Y(1-Y^2)^2\frac{d^2U}{dY^2}+(1-Y^2)^3\frac{d^3U}{dY^3}] \quad (13) \\
 &-kU(1-Y^2)\frac{dU}{dY}-kV(1-Y^2)\frac{dV}{dY}=0
 \end{aligned}$$

while the second equation in (12) reduces to

$$\begin{aligned}
 &-\lambda(1-Y^2)\frac{dV}{dY}+2k^3[2(1-Y^2)(3Y^2-1)\frac{dV}{dY} \\
 &-6Y(1-Y^2)^2\frac{d^2V}{dY^2}+(1-Y^2)^3\frac{d^3V}{dY^3}] \quad (14) \\
 &+kU(1-Y^2)\frac{dV}{dY}=0
 \end{aligned}$$

Now, to determine the parameters m and n , we balance the linear term of highest-order with the highest order nonlinear terms. So, in (13) we balance U''' with VV' to obtain

$$6+m-3=n+2+n-1 \Rightarrow m+2=2n$$

while in (14) we balance V''' with UV' to obtain

$$6+n-3=m+2+n-1 \Rightarrow m=n=2$$

The tanh method admits the use of the finite expansion for both:

$$u(x,t)=U(Y)=a_0+a_1Y+a_2Y^2, \quad a_2 \neq 0 \quad (15)$$

$$v(x,t)=V(Y)=b_0+b_1Y+b_2Y^2, \quad b_2 \neq 0 \quad (16)$$

Substituting $U, U', U'', U''', V, V', V''$, in (13), then equating the coefficient of Y^i , $i=0,1,2,3,4,5$ leads to the following nonlinear system of algebraic equations

$$\begin{aligned}
 Y^0 : &2a_1k^3-a_0a_1k-a_1\lambda-b_0b_1k=0 \\
 Y^1 : &-k(2a_2(a_0-8k^2)+a_1^2+b_1^2+2b_0b_2)-2a_2\lambda=0 \\
 Y^2 : &a_1(-3a_2k-8k^3+\lambda)+a_0a_1k+b_1(b_0-3b_2)k=0 \\
 Y^3 : &k(-2a_2(a_2+20k^2)+a_1^2+2a_0a_2+b_1^2+2(b_0-b_2)b_2) \\
 &+2a_2\lambda=0, \\
 Y^4 : &3k(a_1(a_2+2k^2)+b_1b_2)=0, \\
 Y^5 : &2k(12a_2k^2+a_2^2+b_2^2)=0.
 \end{aligned} \quad (17)$$

Substituting $U, U', U'', U''', V, V', V''$, in (14), then equating the coefficient of Y^i , $i=0,1,2,3,4,5$ leads to the following nonlinear system of algebraic equations

$$\begin{aligned}
 Y^0 : &b_1(a_0k-4k^3-\lambda)=0, \\
 Y^1 : &2b_2(a_0k-16k^3-\lambda)+a_1b_1k=0, \\
 Y^2 : &-a_0b_1k+a_2b_1k+2a_1b_2k+16b_1k^3+b_1\lambda=0, \quad (18) \\
 Y^3 : &2b_2(-a_0k+a_2k+40k^3+\lambda)-a_1b_1k=0, \\
 Y^4 : &-k(a_2b_1+2a_1b_2+12b_1k^2)=0, \\
 Y^5 : &-2b_2k(a_2+24k^2)=0.
 \end{aligned}$$

Solving the nonlinear systems of equations (17) and (18) we get two solution sets

$$\begin{aligned}
 a_0 &= \frac{16k^3+\lambda}{k}, \quad a_1=0, \quad a_2=-24k^2, \quad (19) \\
 b_0 &= \pm \frac{2i\sqrt{2}(4k^3+\lambda)}{k}, \quad b_1=0, \quad b_2=\mp 12i\sqrt{2}k^2
 \end{aligned}$$

and

$$\begin{aligned}
 a_0 &= \frac{4k^3+\lambda}{k}, \quad a_1=0, \quad a_2=-12k^2, \quad (20) \\
 b_0 &= 0, \quad b_1=\pm 4\sqrt{3k\lambda-6k^4}, \quad b_2=0.
 \end{aligned}$$

Case I. From (19) we get the following solution of the system (8):

$$u(x,t)=\frac{16k^3+\lambda}{k}-24k^2 \tanh^2(kx-\lambda t) \quad (21)$$

$$\begin{aligned}
 v(x,t) &= \pm \left[2i\sqrt{2}(4k^3+\lambda)/k \right. \\
 &\quad \left. -12i\sqrt{2}k^2 \tanh^2(kx-\lambda t) \right] \quad (22)
 \end{aligned}$$

Values k and λ can be any real numbers. For example, if we choose $k=\frac{1}{2}$ and $\lambda=-\frac{1}{2}$, the solution (21)-(22) has the form

$$\begin{aligned}
 u(x,t) &= 3-6 \tanh^2\left(\frac{x+t}{2}\right), \\
 v(x,t) &= -3i\sqrt{2} \tanh^2\left(\frac{x+t}{2}\right),
 \end{aligned} \quad (23)$$

Figures 1, 2 illustrate $u(x,t)$ and $v(x,t)$ respectively in the region $-10 \leq x \leq 10$, and $-5 \leq t \leq 5$.

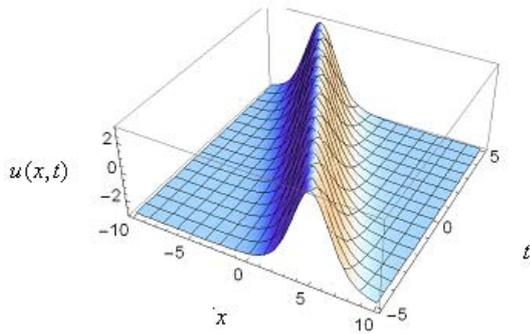


Fig. 1. illustrates $u(x,t)$ in the region $-10 \leq x \leq 10$, and $-5 \leq t \leq 5$

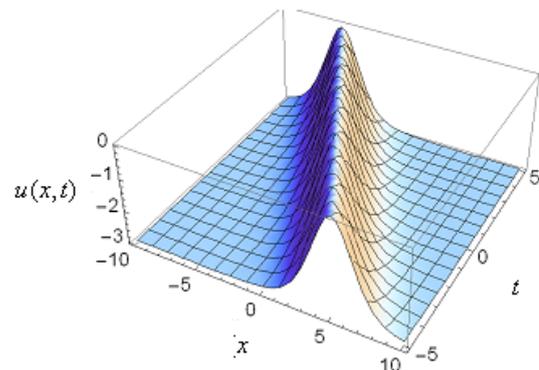


Fig. 3. illustrates $u(x,t)$ in the region $-10 \leq x \leq 10$, and $-5 \leq t \leq 5$

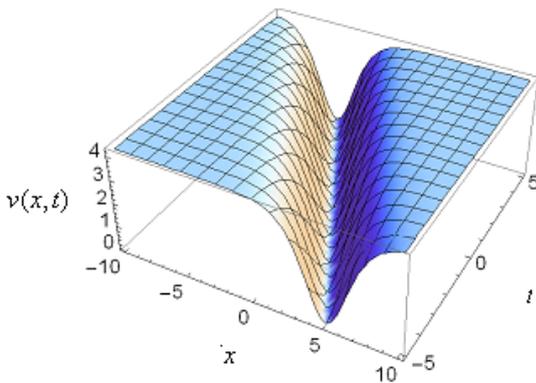


Fig. 2. illustrates $v(x,t)$ in the region $-10 \leq x \leq 10$, and $-5 \leq t \leq 5$

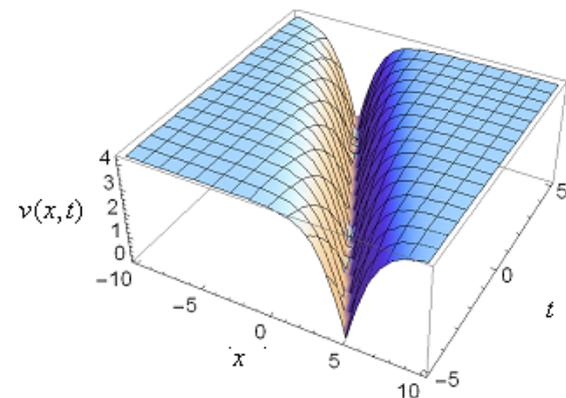


Fig. 4. illustrates $v(x,t)$ in the region $-10 \leq x \leq 10$, and $-5 \leq t \leq 5$

Case II. From (20) we obtain the following solution

$$u(x,t) = \frac{4k^3 + \lambda}{k} - 12k^2 \tanh^2(kx - \lambda t), \tag{24}$$

$$v(x,t) = \pm 4\sqrt{3k\lambda - 6k^4} \tanh(kx - \lambda t).$$

Similarly, by taking $k = \frac{1}{2}$ and $\lambda = -\frac{1}{2}$, we obtain

$$u(x,t) = -3 \tanh^2\left(\frac{x+t}{2}\right),$$

$$v(x,t) = -3i\sqrt{2} \tanh\left(\frac{x+t}{2}\right).$$

Figures 3, 4 illustrate $u(x,t)$ and $v(x,t)$ respectively, in the region $-10 \leq x \leq 10$ and $-5 \leq t \leq 5$.

Example 2. Consider a two component evolutionary system of KdV equations of order 3 (type II) [9]:

$$u_t - u_{xxx} - 2vu_x - uv_x = 0,$$

$$v_t - uv_x = 0. \tag{25}$$

Using the traveling wave transformations in (9), (10) and (11), the nonlinear system of partial differential equations (25) is carried to a system of ordinary differential equations

$$-\lambda U' - k^3 U''' - 2kVU' - kUV' = 0,$$

$$-\lambda V' - kUU' = 0. \tag{26}$$

By postulating the tanh series, and using (10)-(11), the first equation in (26) reduces to

$$\begin{aligned}
 &-\lambda(1-Y^2)\frac{dU}{dY}-k^3[2(1-Y^2)(3Y^2-1)\frac{dU}{dY} \\
 &-6Y(1-Y^2)^2\frac{d^2U}{dY^2}+(1-Y^2)^3\frac{d^3U}{dY^3}] \quad (27) \\
 &-2kV(1-Y^2)\frac{dU}{dY}-kU(1-Y^2)\frac{dV}{dY}=0
 \end{aligned}$$

while the second equation in (26) reduces to

$$-\lambda(1-Y^2)\frac{dV}{dY}-kU(1-Y^2)\frac{dU}{dY}=0 \quad (28)$$

Now, to determine the parameters m and n , we balance the linear term of highest-order with the highest order nonlinear terms. So, in (27) we balance U''' with UV' to obtain

$$6+m-3=m+2+n-1 \Rightarrow n=2,$$

while in (28) we balance V' with UU' to obtain

$$2+n-1=m+2+m-1 \Rightarrow m=1.$$

The tanh method admits the use of the finite expansion for both:

$$u(x,t)=U(Y)=a_0+a_1Y, \quad a_1 \neq 0 \quad (29)$$

$$v(x,t)=V(Y)=b_0+b_1Y+b_2Y^2, \quad b_2 \neq 0 \quad (30)$$

Substituting U, U', U''', V, V' in (27), then equating the coefficient of $Y^i, i=0,1,2,3,4,5$ leads to the following nonlinear system of algebraic equations

$$\begin{aligned}
 Y^0 : &a_1(-2b_0k+2k^3-\lambda)-a_0b_1k=0, \\
 Y^1 : &2a_2(-2b_0k+8k^3-\lambda)-k(3a_1b_1+2a_0b_2)=0, \\
 Y^2 : &a_1(2b_0k-4b_2k-8k^3+\lambda)+(a_0-5a_2)b_1k=0, \\
 Y^3 : &2a_2(2b_0k-3b_2k-20k^3+\lambda)+k(3a_1b_1+2a_0b_2)=0, \\
 Y^4 : &k(a_1(4b_2+6k^2)+5a_2b_1)=0, \\
 Y^5 : &6a_2k(b_2+4k^2)=0.
 \end{aligned} \quad (31)$$

Substituting $U, U', U'', U''', V, V', V''',$ in (28), then equating the coefficient of $Y^i, i=0,1,2,3,4,5$ leads to the following nonlinear system of algebraic equations

$$\begin{aligned}
 Y^0 : &a_0a_1(-k)-b_1\lambda=0, \\
 Y^1 : &(a_1^2+2a_0a_2)(-k)-2b_2\lambda=0, \\
 Y^2 : &a_1(a_0-3a_2)k+b_1\lambda=0, \\
 Y^3 : &a_1^2k+2(a_0-a_2)a_2k+2b_2\lambda=0, \\
 Y^4 : &3a_1a_2k=0, \\
 Y^5 : &2a_2^2k=0.
 \end{aligned} \quad (32)$$

Solving the nonlinear systems of equations (31) and (32) we get:

$$\begin{aligned}
 a_0 &= 0, \quad a_1 = \pm\sqrt{3k\lambda}, \quad a_2 = 0, \\
 b_0 &= \frac{-\lambda+2k^3}{2k}, \quad b_1 = 0, \quad b_2 = -\frac{3k^2}{2}.
 \end{aligned}$$

By replacing these, we obtain the following expressions for $u(x,t)$ and $v(x,t)$ respectively:

$$u(x,t)=\pm\sqrt{3k\lambda}\tanh(kx-\lambda t), \quad (33)$$

$$v(x,t)=\frac{-\lambda+2k^3}{2k}-\frac{3k^2}{2}\tanh^2(kx-\lambda t). \quad (34)$$

Taking $k = \frac{1}{\sqrt{3}}$ and $\lambda = \frac{1}{\sqrt{3}}$, we get

$$u(x,t)=\pm\tanh\left(\frac{x-t}{\sqrt{3}}\right), \quad (35)$$

$$v(x,t)=\frac{1}{6}-\frac{1}{2}\tanh^2\left(\frac{x-t}{\sqrt{3}}\right). \quad (36)$$

Figures 5, 6 illustrate $u(x,t)$ in (35) and $v(x,t)$ in (36) respectively in the region $-10 \leq x \leq 10$ and $-5 \leq t \leq 5$.

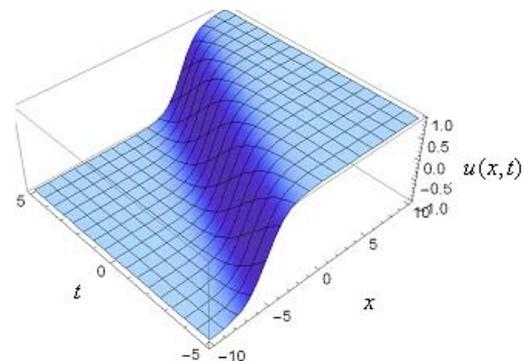


Fig. 5. illustrates $u(x,t)$ in the region $-10 \leq x \leq 10$, and $-5 \leq t \leq 5$

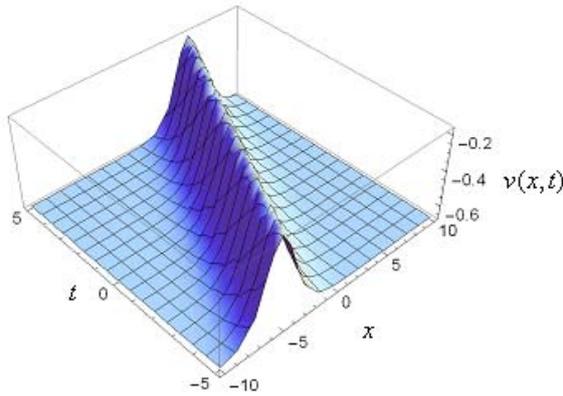


Fig. 6. illustrates $v(x,t)$ in the region $-10 \leq x \leq 10$, and $-5 \leq t \leq 5$

4. Traveling wave solutions

In this section, the traveling wave hypothesis will be used to carry out the integration of the nonlinear wave equation in Example 2 in a generalized setting. Here the power law nonlinearity will be taken into account. Therefore, equation (25) is rewritten with arbitrary coefficients and power law nonlinearity parameter n as

$$q_t + arq_x + bqr_x + cq_{xxx} = 0, \tag{37}$$

$$r_t + kq^n q_x = 0. \tag{38}$$

Here, in equations (37) and (38), a, b, c and k are constant coefficients while n is the power law nonlinearity parameter. In order to integrate (37) and (38), the following traveling wave hypothesis is assumed

$$q(x,t) = g(x-vt) \tag{39}$$

and

$$r(x,t) = h(x-vt). \tag{40}$$

Here, in (39) and (40), g and h represent the nonlinear wave form of the solutions and v represents the soliton velocity. Substituting (39) and (40) into (37) and (38) leads to the ODEs

$$vg' - ahg' - bgh' - cg''' = 0 \tag{41}$$

and

$$vh' - kg^n g' = 0. \tag{42}$$

In (41) and (42) g' represents dg/ds , g''

represents d^2g/ds^2 and so on, with a similar notation for the h variable where

$$s = x - vt. \tag{43}$$

Integrating (42) with respect to s and substituting h in terms of g into (41) leads to

$$vg - \left\{ \frac{ak}{(n+1)v} + \frac{bk}{v} \right\} \frac{g^{n+2}}{n+2} - cg'' = 0 \tag{44}$$

after integration, where in both cases the integration constant is taken to be zero, since the search is for a soliton solution. Now, multiplying both sides of (44) by g' and integrating again, still taking the integration constant to be zero leads to

$$\begin{aligned} (g')^2 &= \frac{2k(a+b+bn)}{(n+1)(n+2)(n+3)cv} \\ &\times g^2 \left\{ \frac{(n+1)(n+2)(n+3)v^2}{2k(a+b+nb)} - g^{n+1} \right\} \end{aligned} \tag{45}$$

which upon integration yields the 1-soliton solution for $q(x,t)$ as

$$q(x,t) = g(x-vt) = A_1 \operatorname{sech}^{\frac{2}{n+1}}[B(x-vt)] \tag{46}$$

where the amplitude A_1 and the width B are respectively given by

$$A_1 = \left[\frac{(n+1)(n+2)(n+3)v^2}{2k(a+b+nb)} \right]^{\frac{n+1}{2}} \tag{47}$$

and

$$B = \frac{n+1}{2} \sqrt{\frac{v}{c}}. \tag{48}$$

These relations pose the constraints

$$cv > 0 \tag{49}$$

and

$$kv^2(a+b+nb) > 0 \tag{50}$$

for odd values of n . Finally, the 1-soliton solution for the r -variable is obtained from (42) as

$$r(x,t) = A_2 \operatorname{sech}^2[B(x-vt)] \tag{51}$$

where the amplitude A_2 in this case is given by

$$A_2 = \frac{(n+2)(n+3)v}{2(a+b+nb)}. \quad (52)$$

Consider again the special case of the system (37)-(38), considered in Example 2 of the previous section (equations (25)). It is obtained for the following values of parameters

$$a = -2, \quad b = c = k = -1, \quad n = 1$$

and is given by

$$\begin{aligned} q(x,t) &= \sqrt{3} |v| \operatorname{sech}(\sqrt{-v}(x-tv)), \\ r(x,t) &= -\frac{3}{2}v \operatorname{sech}^2(\sqrt{-v}(x-tv)). \end{aligned} \quad (53)$$

Taking $v = -1$ we obtain

$$\begin{aligned} q(x,t) &= \sqrt{3} \operatorname{sech}(t+x), \\ r(x,t) &= \frac{3}{2} \operatorname{sech}^2(t+x). \end{aligned} \quad (54)$$

Figures 7, 8 illustrate $q(x,t)$ and $r(x,t)$ in (54) in the region $-3 \leq x \leq 3$ and $-1 \leq t \leq 1$.

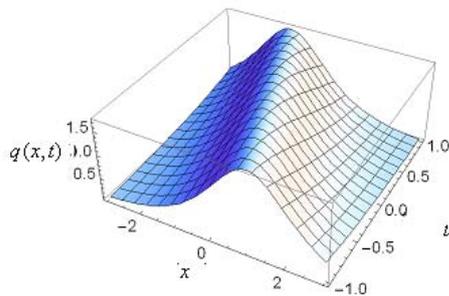


Fig. 7. Illustrates $q(x,t)$ in (54) in the region $-3 \leq x \leq 3$ and $-1 \leq t \leq 1$

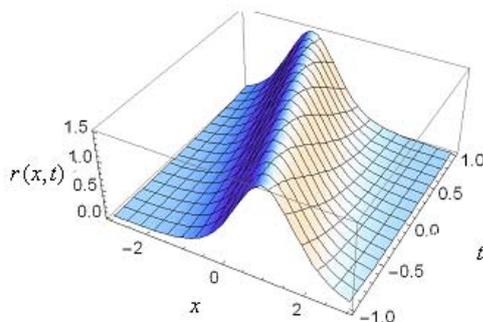


Fig. 8. Illustrates $r(x,t)$ in (54) in the region $-3 \leq x \leq 3$ and $-1 \leq t \leq 1$

Note that the solution (33)-(34) obtained in the

previous section differs from the general solution (equations (46) and (51)) derived in this section. This is because the expression (46) for $q(x,t)$ does not have finite tanh expansion, which was the assumption in the previous section.

5. Conclusions

In this paper, we applied tanh method for solving nonlinear coupled partial differential equations. The tanh method requires transformation formulas and traveling wave solutions were obtained. It is observed that all solutions of the coupled KdV systems type I, II obtained in this paper by tanh method were successfully compatible to the results obtained by Marwan [9] using the HPM. It is also shown that using traveling wave hypothesis, one can obtain sech solution of type II coupled KdV system, in more general setting.

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